Modelling Investment Strategies: Bayesian Learning, Regime Switches and Evolutionary Finance

By

Martin Anastasov Anastasov

Submitted in accordance with

the requirements for the degree of

Doctor of Philosophy

Centre for Advanced Studies in Finance

Leeds University Business School

University of Leeds

September, 2015

The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

Contents

List of Figures

List of Tables

Abstract

In this research project we endeavour to model a financial marketplace dominated by a few interacting large institutional investors and draw conclusions about the financial market dynamics that this interaction gives rise to. More specifically, we study the problem of institutional investors, such as pension funds and life assurance companies, which operate in an environment of uncertain cash inflows and uncertain payouts with a minimum threshold.

Investment strategies are taken as model primitives and an artificial financial market is populated by multiple investor types. Trading and investment decisions take place in discrete time. There exist a certain predetermined number of long-lived risky assets paying a random amount of dividends at each discrete point in time, as well as a risk-free asset with a constant interest rate. The risky assets generate random dividend intensities. Asset payoffs are aggregated and paid to investors at the end of each time period. The general economy is assumed to follow a hidden Markov model with two states, corresponding to normal and recessionary regimes. The existence of a minimum consumption constraint implies the possibility of bankruptcy. The effect of this occurrence on market clearing is modelled explicitly.

We solve our model by means of numerical simulation. The size of the wealth endowment of the different agents is monitored through time over a large number of simulation runs. Our results suggest that both trend following and value investing strategies can be selected by the market under different circumstances. These two results lead to markedly different outcomes for the economy, as the prevalence of the trend following style leads to destabilization of the marketplace, volatility clustering and severe deflationary spirals. Dividend yield and modified regime-switching CAPM strategies are never selected by the market.

Key words: institutional investors; evolutionary finance; hidden Markov models; artificial market; trend following; value investing;

Chapter 1 Introduction

The credit crunch and the economic recession, that most countries have had to grapple with in the last years, have been hotly debated topics, which have been piquing everybody's curiosity. This prolonged period of negative economic growth, rising unemployment, and frequent bankruptcies has baffled professional economists and ordinary people alike. And while the attention of financial specialists has been focused on the more pressing global issues, like how to avoid large-scale job losses, and how to prevent the economy from plunging into an even deeper recession, there are also some extremely important issues that have not been receiving the attention they deserve.

The research project outlined in this document addresses one of them - the pensions crisis. The pensions crisis refers to the difficulties faced by some institutional investors, such as pension funds and life insurance companies, which unfortunately have also impacted a number of large companies directly related to them. The effect of these occurrences has been felt by ordinary people as well, particularly by those, who are due to retire in the near future.

The pensions crisis is a direct consequence of the global financial crisis. With the worsening of economic conditions, investors and banks became more cautious and nearly ceased lending to each other. Overall economic activity slowed down, profit margins were squeezed and doing business became more difficult. Faced with financial problems and encouraged by a wide-spread lack of confidence, investors started gradually withdrawing from financial markets. This naturally brought about a prolonged slump in equity prices. This in turn negatively affected the funding levels of pension funds and insurance companies.

In the United Kingdom, the Financial Conduct Authority (FCA) regulates funding levels and capital adequacy by means of the Individual Capital Adequacy Standards (ICAS). Additional regulation is imposed by European Union supervision in the form of the Solvency II standards, which are also in effect within the jurisdiction of the United Kingdom.

The depreciation of equity prices impacted adversely all long-term investors, who still had to meet some predetermined regular liabilities, stemming from annuity contracts they had signed in previous years, regardless of the economic situation. Pension funds saw their investments rapidly depreciating. By itself, this would not have been such a severe problem, since a loss is only realized when an investment is liquidated. Institutional investors, however, did post negative results in the form of actuarial losses - accountants and actuaries regularly review a pension plan's investments and mark them to market to reflect more accurately the current status of the institution's funding balance. This measure was adopted to help protect plan members by preventing the accrual of a large mismatch between the present value of future pension liabilities and the value of available assets.

Despite its good intentions, however, marking to market actually harmed plan members, instead of protecting them. This is so because of certain accounting and regulatory requirements, which demand that a sponsoring company should eliminate any deficits in its pension plan if the ratio between assets and liabilities falls below a predetermined threshold, called the minimum funding ratio. In the prevailing tough economic conditions, these firms had already been facing serious financial problems, due to loss of clients, reduced profit margins, and disruption of their working capital cycles. When the burden of extinguishing pension plan deficits was added to this, some companies went bankrupt, while others ceased all discretionary cash outflows, including the payment of dividends.

This in turn fed back very negatively to pension funds and insurance companies, since many of them were invested in these big corporations. When current income in the form of dividends diminished, institutional investors had no choice but to liquidate part of their investments, so as to be able to meet their liabilities. Since such extreme conditions as prolonged periods of negative growth affect all major players in the economy, they are a source of systematic risk. This meant that a lot of large pension funds and insurance companies were negatively impacted at the same time. The result of this has been a mass sell-off of institutional asset holdings, which brought about a further decline in asset values, thus reinforcing the vicious deflationary circle.

Based on all of the above, the main ambitions of our research project are as follows. Firstly, to examine this two-way link between the investment and risk management decisions of large institutional investors and the dynamics of asset prices. Secondly, and perhaps more importantly, we also hope to be able to recommend an optimal policy for asset allocation and risk management of pension funds and insurance companies, such that it will be possible for an institutional investor with periodic liabilities to avoid a situation, which forces him to liquidate investments at unfavourable prices, and in the same time allows him to retain enough exposure to the upside potential of being invested in risky assets.

If these objectives are achieved, the contribution of this research project will be two-fold. Firstly, it will provide important extensions to existing theoretical literature dealing with intertemporal investment-consumption decisions, multi-period portfolio choice and asset allocation, and evolutionary market selection. Secondly, the obtained results from the theoretical models, that we implement, can serve as a basis for subsequent empirical studies. Once the various model parameters have been calibrated to fit past data, the theoretical relationships that we find can then be used in practice by pension fund trustees and managers to help them make better asset allocation and risk management decisions.

The rest of this document is structured as follows. Chapter 2 presents existing literature from different research areas dealing with the problem of multi-period optimal asset allocation with minimum consumption constraints. Chapter 3 presents the theory behind, and all components of the model we employ to study the above mentioned problem, as well as its practical implementation. Chapter 4 collects the results and analyzes them. Finally, chapter 5 summarizes our findings and concludes.

Chapter 2 Literature Review

Contents

2.1 Contractual Savings Institutional Investors: General Background and Regulatory Framework

We begin this section by providing some background information on institutional investors, their idiosyncratic characteristics, as well as a brief guide to the regulatory framework in which they operate.

The types of financial institutions, with which we will be dealing in this paper, arose from the need for efficient fiduciary management, which is generally defined as a way of pooling funds to form sizable investment portfolios and the organization of their investment management process (Nunen (2008)). Even though initially every person in employment was responsible for managing their own savings and investments, with the increasing complexity of capital markets and transaction costs, it soon became clear that large specialized financial institutions that manage savings collectively on behalf of small investors with specific risk, return and maturity objectives were needed. These entities are collectively known as institutional investors (Davis & Steil (2004)). Pension funds, insurance companies, unit trusts, hedge funds and mutual funds are all classified as institutional investors. What makes these entities attractive is their ability to provide risk-pooling and diversification benefits to the individual retail investor, which would have been difficult to realize if they invested in the market on their own. Other useful features of institutional investors include the provision of liquidity to capital markets, their expertise at efficiently absorbing and analyzing relevant information, as well as their ability to match the maturity profiles of assets and liabilities. Of course, being a large entity allows institutional investors to achieve economies of scale in investing, enables them to participate in large, indivisible investments, and reduce both transaction costs for execution, and the fixed costs entailed in gathering and analyzing information.

From the perspective of financial intermediation, institutional investors are not unlike other big financial institutions, such as banks for instance, in providing valuable functions for the economy. Davis (1996) mentions a number of those functions, such as clearing and settling payments, pooling of funds, transferring economic resources, managing uncertainty and controlling risk, use of price information, as well as dealing with incentive problems. In an idealized Arrow-Debreu world where markets are efficient and frictionless, and there are no transaction costs, it is not trivial to justify the existence of financial intermediaries. In terms of value creation in the real world, however, Davis (2000) argues that market frictions such as asymmetric information, transaction costs and credit rationing make a financial structure based on intermediation an indispensable tool for achieving a Pareto-optimal allocation of resources. From the aforementioned types of institutional investors we have chosen to limit ourselves to the study of pension funds and insurance companies for reasons that will be disclosed later in this document. In what follows we provide some factual information on these financial institutions, paying special attention to some peculiarities in the way they operate.

2.1.1 Pension Funds

Davis (2000) defines pension funds as "a form of institutional investor, which collect, pool and invest funds contributed by sponsors and beneficiaries to provide for the future pension entitlements of beneficiaries". The notion of saving during one's working life (the accumulation phase) to provide financially for the period of retirement is a relatively recent one and came about due to improved quality of life and longer life expectancy in some countries. The formal beginning a social security system may be traced back to 1881 when Otto von Bismarck enacted an arrangement for the provision of financial protection against the income consequences of old age and ill health (Nunen (2008)). Pension funds are typically sponsored by employers, even though employees are also expected to contribute, usually in the form of additional top-up contributions.

2.1.1.1 Funded vs. Unfunded Schemes

Pensions may be financed out of a reserve fund built up by accumulating contributions during a person's working life and investing them in earning assets. This is known as accumulation of funds or a funded pension scheme (Blake (2003)). Alternatively, the pensions of the retirees could be paid entirely out of current contributions by both younger employees, who are still active in the labour market, and employers. In return for their contributions, employees receive the promise that their own pensions will be paid in the same manner by the next generation of workers. This system of pension funding is known as pay-as-you-go (PAYG). Historically, the economy of Anglo-Saxon countries has been structured around well developed capital markets. People's confidence in such a market-based system has contributed to financial markets being seen not only as a means of channelling resources to their most productive use, but as a store of value as well. Traditionally, those countries have exhibited a preference towards funded schemes, while PAYG schemes have been popular in bank-dominated and centrally-planned economies. The demographic trend of ageing population, as well as the potential for increased efficiency, has caused most occupational pension schemes today to move to funded plans. The accumulated funds can be managed either in-house by the sponsoring company, or outsourced to a specialized pension fund or insurance company.

2.1.1.2 Defined Benefit vs. Defined Contribution

While the principal objective of a funded plan is to accumulate enough assets from contributions and investment income to ensure they are sufficient to meet pension liabilities in the future, the actual mechanics of how this happens can differ. There exist two main varieties of funded pension schemes - defined-benefit (DB) and defined-contribution (DC) (Blake (2006)). The characteristics of each have important implications for the investment objectives and incentives of both parties - sponsoring companies and beneficiaries. In both schemes the employer and beneficiaries make fund contributions to be invested in earning assets. The value of the terminal accumulated wealth available to be paid out as pensions depends on the investment performance of the assets, in which the contributions are invested. In a defined contribution scheme the funds are held in trust for the employee in a tax-deferred retirement savings account (e.g. IRA accounts in the United States). Contributions are usually calculated as a predetermined percentage of salary, which needs not necessarily be constant over one's career. Accrued investment income is usually tax-free for the period during which the funds remain invested in the trust account and at retirement the employee may opt to receive either a lump-sum payment or an annuity (Bodie (1990)). The investment risk is entirely borne by the beneficiary - the responsibility of the sponsoring company is exhausted with paying the agreed contributions, and this is where the name of the scheme is derived from.

Conversely, in a defined-benefit scheme the beneficiary knows the minimum amount he can expect to receive upon retirement. The latter is calculated based on the employee's history of service and salary. The minimum promised annuity is the employer's liability and hence the name "defined-benefit". While there also exist average salary schemes, the most popular defined-benefit plans have been found out to be the final salary schemes, which are back loaded in the sense that the present value of benefits earned each year is greater in the later years of the plan. This is so owing to a number of actuarial assumptions used to calculate pension liabilities for the sponsoring company. One of these is the average rate of forecasted inflation, at which the employee's salary is assumed to grow annually and the other is the chosen discount rate to reflect the time value of money (Blake (2006)).

There is a variety of opinions as to the true economic meaning of the word "pension". Some authors consider it as insurance on retirement income (e.g. Bodie (1989)). Others view it in the light of an option on pension fund assets (Blake (1998)). Even though these different connotations can change the analytical framework for modeling pension problems, a pension essentially represents an entitlement to an annuity of payments in the future. Bodie (1990) presents two kinds of annuities - fixed and variable, with the former being much more popular in the pension fund and insurance industry. A DB plan, however, should be viewed not just as an ordinary, fixed annuity, but rather as a variable annuity with a fixed minimum (Bodie (1990)). The sponsoring company is free to increase the amount to be paid out depending on its financial strength, the increase in the living costs of retirees or the investment performance of plan assets.

Curiously, the benefit from investment performance of the plan assets is actually asymmetric because the sponsoring company does not always appropriate the full benefit of good fund performance. However, it is obligated to provide a fixed floor and thus suffers to the full extent in the case of sub-par investment results. The latter occurs mainly due to the accounting and regulatory framework, in which pension funds operate, which postulates that the sponsoring company recognize an unfunded liability on its balance sheet but forbids the inclusion of any overfunding in the financial statements (e.g. FASB statement 87 in the US; International Financial Reporting Standards (IFRS) require a reconciliation of the plan funded status, so that only the recognized portion of actuarial gains or losses are included on the financial statements). Without this asymmetry, Blake (2006) contends that a DB plan can be analyzed as a DC plan, in which the sponsoring company holds a call option on the fund's assets while the beneficiaries hold a put option. Even with the reported asymmetry, however, the sponsoring company usually has a considerable interest in the upside potential, since under most jurisdictions employers are required to eliminate any surpluses through contributions "holidays".

The unique characteristics of DB and DC schemes impinge on the fund's investment behaviour. The main emphasis in a DC scheme is to achieve efficient diversification and thus to maximize expected return for a given level of risk, while the investment policy of DB schemes focuses on sound asset-liability management and actively uses fixed income instruments matching the duration of the liabilities. In terms of investment style, most authors suggest three main approaches to pension fund management: index tracking, active management, and asset liability management (Blake (2006)). The chosen investment approach has major implications for asset allocation. These will be discussed below.

The majority of funded plans nowadays are of the defined-benefit type but due to the recent difficulties that sponsoring companies have been facing with their pension schemes, and the fact that defined-contribution schemes are much more portable between jobs, there has been a substantial move worldwide towards DC plans. Still, with the sizable amount of assets already accumulated in DB schemes, they will continue to represent the greater percentage of funded schemes for the foreseeable future. Additionally, through their much more tangible impact on sponsoring companies, they have a much more noticeable influence on economies. For these reasons we elect to focus on funded schemes of the defined benefit type in the research project outlined below.

2.1.1.3 Regulation

Pension funds and other institutional investors face a number of risks, such as longevity risk, inflation risk, investment risk, the risk of default of the sponsoring company, and others. Following the recent precipitous fall in equity prices worldwide and the implications this has had on institutional investors, we focus exclusively on investment risk in the current project.

To adequately address all those risks, a substantial amount of regulatory legislation is necessary. In the United Kingdom pension funds are mainly regulated by the Finance Act (1986 and subsequent revisions), the Pensions Act (1995 and subsequent amendments), the Trustee Act (2000), the Social Security Act (1990), and to an extent, the Myners' Report "Institutional Investment in the United Kingdom" (2001). All these statutes, however, tend to be non-prescriptive and fund trustees are granted considerable leeway in making and implementing decisions, which they consider to be for the benefit of fund members. This governance principle of acting rationally and in the best interest of stakeholders was named "the prudent investor rule" in the Trustee Act (2000). One of the desired outcomes of the current research project is to aid trustees in their decisions related to asset allocation and portfolio insurance by providing investment behaviour guidelines by means of a theoretical framework of stock market dynamics with the presence of large institutional investors and price impact.

In the United States, arguably the single most important piece of legislation concerning the operation of pension funds is the Employee Retirement and Income Security Act (ERISA) from 1974, which created the Pension Benefit Guaranty Corporation (PBGC). In 1978 the popular 401(k) plans were introduced, which quickly gained popularity and became one of the main driving forces behind the move towards defined-contribution schemes.

2.1.2 Insurance Companies

Insurance companies are an eclectic group of enterprises, whose common feature is the provision of insurance and investment services to retail investors. Dorfman (2004) defines insurance as "a financial arrangement that redistributes the costs of unexpected losses". The general idea is for a financial intermediary to pool assets from many beneficiaries, just as in the case of pension funds. These, however, are to be used to pay indemnities to any of the beneficiaries who happen to suffer a loss while insured. In this way, instead of facing the loss on their own, the claimant's indemnity is paid out of the pool of funds accumulated from insurance premiums. Correct contract pricing and analysis of all potential risks are essential for any insurance company in order for it to be able to realize consistent profits over time, regardless of the realized states of nature.

Based on what constitutes an insurable event, insurance can be decomposed in several branches: for example, fire insurance, business income coverage, marine insurance, casualty insurance, credit insurance, life insurance etc. (Dorfman (2004)). Since in this research project we are mainly interested in how various investment and risk management decisions of institutional investors impact the overall financial market dynamics, we concentrate exclusively on the branch of life insurance for reasons that will become apparent below.

Life insurance, as the name suggests, is based on human life contingencies and comes in two forms. If the insurable event is the risk of premature death then the contract is known merely as life insurance. Under the arrangements of such contracts, the immediate family of the beneficiary receive predetermined lump-sum compensation in the event of the beneficiary's untimely death. Conversely, in case the insurable event is extended lifetime, the contract is known as an annuity. Under the arrangements of an annuity, the beneficiary is promised a stream of regular payments until he dies. Insurance companies finance this obligation by pooling premium income from many individuals and investing it in earning assets, just like a pension fund would do. Additional help in meeting their obligations is provided by the law of large numbers: the premium income from some of the beneficiaries who die too early to claim any benefits gets redistributed to claiming beneficiaries. A detailed exposition of the various kinds of available contracts, such as whole-life, universal life, variable life, or endowment life insurance, is presented in Vaughan & Vaughan (2002).

In terms of regulation, the most important statutory documents for insurance companies in the United States are the McCarran-Ferguson Act (Public Law 15) and more recently the Gramm-Leach-Bliley Act (1999) (Graham & Xie (2007)), while in the United Kingdom they are the Insurance Companies Act (1982), and more generally the Financial Services Act (1986) and the Financial Services and Markets Act (2000) (Hardwick & Guirguis (2007)).

The mechanics of annuity contracts are very similar to the arrangements under most of the pension fund schemes discussed in the previous section. In fact, the two forms of savings are complementary - individuals often choose to have life insurance on top of their pension savings so as to eliminate their longevity risk, i.e. exhausting their pension income in their lifetimes (for other motivations to save in the form of life insurance see e.g. Dorfman (2004)). Collectively, pension funds and life insurance companies are known as contractual savings institutional investors due to the special contractual annuity guarantee feature of their liabilities. The fixed minimum level of their obligations makes them very interesting to analyze in the wake of the current financial crisis and plummeting stock markets. While other institutional investors have long-term investment horizon and are better positioned to survive the recent severe slump in asset values and current income, pension funds in their mature stage and life insurance companies have been seriously affected, so much so that this occurrence warrants further investigation.

2.2 Why Institutional Investors

The motivation for specifically focusing our analysis on institutional investors is simple: their sheer size and the number of participating plan members makes them important enough to have a significant influence on both the macro level (economy, sponsoring corporations and financial markets) and the micro level though the numerous ordinary employees who depend on them for their financial well-being during old age.

2.2.1 Institutional Investors, Capital Markets and Economic Growth: Theory

Apart from their sheer size, institutional investors are important for the economy in other ways. On a more general level, Levine (2005) provides a comprehensive survey of the theoretical and empirical work that has been carried out on the connection between the efficient functioning of the financial system and economic growth. Despite the controversy of this issue, the general opinion is that financial systems contribute to economic growth by lowering the cost of researching potential investments; exerting corporate governance; trading, diversifying and managing risk; mobilizing and pooling savings; conducting exchanges of goods and services; and mitigating the negative consequences that random shocks can have on capital investment. As the most sizable participants in the financial system, institutional investors and banks clearly perform a major share of these functions and thus contribute to economic growth.

In addition to contributing towards productivity and economic growth, it has been suggested that institutional investors also promote the development of capital markets and financial innovation. This argument, however, has sparked considerable controversy over the causality of the relationship. Vittas (1998) tries to find a middle ground, and essentially confirms the arguments in the above paragraph, by arguing that while the promotion of pension funds and insurance companies can help in the development of capital markets and should be pursued, mutual funds are unlikely to be as successful in the environment of an insufficiently mature financial system.

More narrowly, focusing solely on insurance, specifically in emerging markets, Skipper (1997) argues that insurance companies can contribute to a country's economic development in at least seven ways: insurance promotes financial stability and reduces anxiety; it can be a substitute for government security; it facilitates trade and commerce; mobilizes national savings; enables risks to be dealt with more efficiently; insurance companies have an economic interest in assisting the insured curtail their losses; and they also foster a more efficient allocation of a country's capital.

2.2.2 Institutional Investors, Capital Markets and Economic Growth: Empirical Evidence

On the empirical side, Harichandra and Thangavelu (2004) examine the impact of institutional investors on stock market development and economic growth using data from the OECD countries. The authors study the problem both at the aggregated and disaggregated level, making use of dynamic panel VAR estimation. The results suggest that institutional investors Granger cause the level of financial sector development and economic growth but this does not hold true for banks. At the disaggregated level, an interesting finding is the result that the development of institutional investors is actually Granger caused by market capitalization, while institutional investors Granger cause liquidity and turnover in financial markets.

Catalan et al. (2000) also provide international evidence supporting the argument that there exists a robust causal relationship between contractual savings institutional investors and stock market development, with Granger causality running from institutions to markets. The authors find this effect to be especially pronounced in countries lacking deeply developed capital markets. In their framework, pension funds and insurance companies add value in terms of improving market capitalization and volume of trades, as well as increasing the pace of financial innovation.

The case for insurance companies alone is not significantly different. Webb et al. (2002) analyze the conjoint effect of the banking system and insurance companies on capital formation and output within the context of a neoclassical Solow-Swan growth model. Using panel data for fifty-five countries over the period 1980-1996, the instrumental variables estimation with three least squares simultaneous equations finds that the activity of banks and insurance companies promotes productivity, with the total effect of the two entities being greater than the sum of their individual effects. Findings of similar positive effects of insurance on economic growth and financial development, but this time using data from developing countries, are presented in Outreville (1990).

Nonetheless, the empirical support for whether insurance companies benefit the economy or not is not as unanimous as it may appear from the above paragraph. In their cointegration analysis using real GDP and total real insurance premiums data from nine OECD countries for the period 1961-1996, Ward & Zurbruegg (2000) provide mixed evidence on the direction of causality in the long-term relationship between insurance activity and growth. The authors contend that the direction of causality must be determined by country-specific factors and specific national circumstances, since they find that for some of the countries, insurance activity Granger causes economic growth, while for the rest, the opposite is true.

Later, however, their findings were refuted by Kugler & Ofoghi (2005) who suggested that the results in Ward & Zurbruegg (2000) might have been biased by an aggregation problem, caused by their choice to use the total value of written insurance premia as one of the variables in their model. Kugler & Ofoghi (2005) study reports a positive long-run relationship between insurance and growth using data from the UK. Moreover, in a more recent paper, Arena

(2008) confirms the findings in Kugler & Ofoghi (2005) using a different econometric methodology (generalized method of moments) with data from both industrialized and developing countries. Arena's (2008) work also provides an important motivation to study financial and insurance markets within a unified framework on a theoretical level.

The empirical pensions literature, albeit of smaller size, tends to be supportive of the economic significance of pension funds as well. Davis & Hu (2004) examine the relationship between the size of pension fund assets and economic growth within a modified Cobb-Douglas production function model and find positive results for both OECD countries and emerging markets, with the effect once again being more pronounced in the developing countries. A different approach is taken by Apilado (1972), who hypothesizes that if pension funds do not divert savings from other traditional means of accomplishing intertemporal transfer of wealth, then they can contribute towards economic growth by increasing the aggregate level of savings available for investment. If this is not true, then the increasing size of pension funds may be a cause for concern, signifying increasing rivalry among financial intermediaries for household deposits. In his empirical study the author is able to confirm his initial hypothesis and concludes that pension funds are capable of enhancing economic growth.

2.3 Price Impact of Institutional Investors

2.3.1 Factors Determining Institutional Price Impact

Initial research in this area sought to identify potential reasons, which caused large institutional investors to have price impact. Chan & Lakonishok (1995) examine all trades executed by thirty-seven large investment management firms during the period 1986-1988 and study the price impact of the entire sequence of trades. The three most important factors determining the magnitude of the price impact turn out to be the capitalization of investment firms, relative size of the sequence of trades, and the identity of the management firm behind the trades. The significance of this study lies in the fact that it explicitly proposes an explanation of what the reasons for institutional price impact may be, even though the paper deals with sequences rather than individual trades. The finding that assets under management and the size of trades are determining factors is not surprising and it coincides with the common intuition shared in the previous paragraph. The third factor, however, signals there might be more to the story than merely size. Financial institutions have a reputation for being sophisticated investors and their behaviour is often used as a signal by other investors and actively emulated.

2.3.1.1 Assets under Management and Size of Trades

There is further empirical and theoretical support for both of these factors causing price impact. With regard to the argument of size, Gompers & Metrick (2001) examine the institutional demand for securities with particular features and how it influences the prices and returns of assets possessing these characteristics. In particular, the authors use the small-firm premium as documented by Banz (1981) as a motivation for the study and go on to hypothesize that the reversal of this effect since 1980 can be explained by the differing investment preferences of large institutional investors. Specifically, Gompers & Metrick (2001) note that such investors generally invest in stocks that are larger, more liquid and have had relatively lower returns in previous years (so called value stocks). In their empirical study the authors use the equity holdings of all institutional investors with at least $100 million at their disposal for investment purposes during the period 1980-1996. The results show that during the period under study the examined institutions approximately doubled their market share and by 1996 they controlled more than half of the equity market. With this, they brought about a steady shift in demand away from growth stocks, which impacted negatively on their price and returns, and in effect reversed the small-size premium.

The study by Gompers & Metrick (2001) is of prime importance because it unambiguously showed that institutional investors were capable of inducing major market shifts through their behaviour, if they acted in a concerted manner. This finding serves as a source of motivation for the endogenous way, in which we specify the asset price formation in our theoretical model described below.

2.3.1.2 Informational Content of Institutional Trades

In terms of the argument of information signalling, an important line of research is the "stealth trading" hypothesis. Chakravarty's (2001) starting point is precisely this theory, according to which informed investors (or those in possession of inside information) will trade gradually with trades of a certain fixed size. The trades should be neither large enough to prematurely release their information content nor too small, since that would be inefficient in terms of transaction costs. The hypothesis then states that one would expect to see a disproportionately large price impact of these medium-sized trades relative to their share of overall trading volume because of the information they will bring to other market participants trying to imitate the behaviour of informed investors.

Chakravarty (2001) decides to test this hypothesis by examining whether the medium-sized trades in his sample are initiated by institutional investors or not. The empirical findings confirm that the medium-sized trades with the most disproportional price impact were indeed initiated by institutions, which are found out to be informed investors. This confirms the findings in previous papers, such as Walther (1997), Sias & Starks (1997), and Badrinath et al. (1995), which claim that institutional investors may differ in their level of sophistication in response to information, which makes them "smart" or "informed" traders.

Like Gompers & Metrick (2001), the study by Chakravarty (2001) also has important general implications. On one hand it confirms that the size of institutions' assets under management or trades is not the only factor that determines their price impact. Indeed, large investors can move the market even with medium-sized trades. On the other hand, this ability testifies to the fact that institutional investors' second means of influencing the market is through the information content of their behaviour, which is not insignificant, especially in times of crisis when investors lack confidence. While the first of our research questions does not consider this mechanism of price impact, the informational content of the behaviour of institutional investors, and how other agents in the market learn from it and change strategies accordingly, is the focal point of our second research question presented below.

2.3.2 Asymmetry of Price Impact

What is more, price impact turns out to be only half of the story. Chiyachantana et al. (2004) find that the influence of institutional investors on asset prices is actually asymmetric. In this paper the authors examine data from 37 countries on institutional trading in international stocks in two separate periods: 1997-1998 and 2000-2001. The reported results suggest that depending on prevailing market conditions, block institutional trades can have different effects. In bullish markets, such as the period 1997-1998, the price impact of purchases is stronger than that of sales. Of particular interest to us is the finding that this asymmetry is reversed in bearish markets, like for example the last stock market crash of 2000-2002, which bears great resemblance to the current market conditions.

These results empirically prove the theoretical hypotheses of an earlier paper by Saar (2001), who presents a case that the trading strategies of institutions create a difference between the informational content of buys and sells. In his probabilistic model the author shows that the magnitude of the asymmetry is determined to a great extent by the history of price performance - the longer the run-up in stock prices, the less the asymmetry. Recent price performance is found out to have the greatest informational content.

2.3.3 Institutional Herding and Positive-Feedback Trading

Asymmetric price impact often compounds other undesirable features of the behaviour of large investment entities. Lakonishok et al. (1992) use data pertaining to the equity holding of 769 tax-exempt institutional investors in an attempt to gain some insight on their potential effect on asset prices. Interestingly, the authors observe two regularities in institutional investment behaviour - herding and positive-feedback trading. The former refers to imitating other investment managers by purchasing or selling the assets the latter have recently traded. Positive-feedback trading is explained as buying past winners and selling past losers. The reason why these phenomena are important for our study is because combining herding and positive-feedback trading with the price impact literature has caused a number of authors to question if institutional investors actually destabilize capital markets by acting in such a collective manner. Such a possibility has been particularly realistic in recent times, in light of our discussion above concerning regulatory requirements that prompt pension funds and life insurance companies to sell part of their assets to maintain a certain asset-liability ratio, and how this might have reinforced the downward spiral in asset values.

The academic literature on this issue tends to examine mostly developing countries, whose less capitalized financial markets tend to be more prone to institutional influence. Aitken (1996) examines the trend of preference towards emerging markets exhibited by institutional investors. This investor sentiment statistically explained the autocorrelation of returns in emerging capital markets during the time, in which large investors were expanding their equity holdings there. In the author's view, this can be seen as a confirmation of the hypothesis that sharp shifts in large investor sentiment are responsible for bubble-like booms and busts, as well as asset price overshooting, like in the case of the Asian crisis, for example.

Indeed, as OECD analysts Reisen & Williamson (1994) also note, with the increasing degree of stock market integration, institutional investors are capable of influencing not only the financial markets in their own countries, but can also start significant worldwide trends. Therefore, preserving macroeconomic stability necessitates careful consideration of capital controls on cross-border institutional investment. In the case of countries with more developed capital markets, however, empirical studies tend to disprove the validity of the destabilization hypothesis: Bohl & Brzeszczynski (2006), for instance, don't find any empirical support using data from Poland.

2.3.4 Illiquidity

Apart from the size of institutional trades and the informational content of their behaviour, the literature also mentions a third factor explaining why asset prices change as a result of the investment activity of pension funds - illiquid markets. Illiquidity has been shown to influence institutional price impact to a much lesser extent (Sias et al. (2001)) and can be considered a subcategory of the size-of-trades argument since in less liquid markets, characterized by longer waiting times between consecutive trades, trades of any size can have an impact due to slower price adjustment (Dufour & Engle 1999). Activity in such markets is slow and since there are fewer investors actively analyzing them and taking appropriate actions to eliminate arbitrage opportunities, any trade can have more than a temporary mispricing effect on prices.

This form of price impact is not unique to institutional investors but since during the pensions crisis of 2007-2008 borrowing, and hence liquidity, were substantially restricted, this might have augmented the two main causes of institutional price impact discussed above. Illiquidity is explicitly taken into consideration in some of the available theoretical models on institutional investors (DeGiorgi (2008), Berry-Stolzle (2008), Vath et al. (2007)) but we elect not to do so, since the focus of our research lies elsewhere and we are content to merely account for price impact in our model specification without pursuing further the various reasons for its occurrence.

2.4 Institutional Asset Allocation

Asset allocation is frequently defined as a systematic way of spreading one's financial resources across different assets in the hope of reducing overall risk by diversifying one's portfolio (Gibson 2008). In the case of pension funds and insurance companies, asset allocation usually means splitting the assets to be invested in shares or bonds, which are the most popular instruments among institutions - the so called "stock-bond mix" (Harrison & Sharpe 1983).

There are different kinds of asset allocation each having a specific investment focus and objective. For instance, strategic asset allocation pertains to the general long-term investment policy; constant weighting asset allocation requires continuous rebalancing of the portfolio as the market values of different asset classes move; tactical asset allocation is a short-term deviation from the strategic investment policy in order to exploit favourable market conditions by temporarily tilting the portfolio towards a particular asset class.

Apart from being an indispensable tool for risk reduction, in institutional investing specifically, asset allocation is an important consideration because of the reliance on benchmarks for the evaluation of investment performance. This is so due to the fact that each asset class has a different exposure to the economy, which is the same for every party invested in this asset, while other factors that have been proposed as determinants of portfolio returns, such as individual skills in market timing and security selection, tend to vary and are never uniform across all market participants. Even though separating and measuring the contribution of each source of portfolio returns to overall performance is a controversial issue, a large number of academic studies show that generally more than ninety percent of the differences in total returns achieved by institutionally managed funds are accounted for by strategic asset allocation decisions rather than market timing or security selection (e.g. Bogle (1994), Brinson et al. (1995)).

Conventional investment wisdom postulates that asset allocation decisions should be based on one's investment horizon. Considering the case of a group of long-term investors, similar in many respects to the case of pension funds and life insurance companies, in which we are interested, Thaler & Williamson (1994) pose the interesting question why should college and university endowment funds not be 100 % invested in stocks? After all, as Bodie (1990) notes, it is a well-established empirical fact that well-diversified portfolios of US stocks have outperformed Treasury bills, bonds and inflation over holding periods longer than twenty years for almost any starting date since 1930.

What is more, Thaler & Williamson (1994) also show that as the investment horizon increases, the probability that stocks will outperform bonds approaches unity. So why then not just be fully invested in equities and forget about tricky asset allocation decisions? According to a number of theoretical papers, which will be discussed in more detail below (e.g. Samuelson (1969), Merton (1969), Merton (1971), Fischer (1983)), the statement that shares are less risky in the long-run is a fallacy. Additionally, there is a potential for shortfalls in any one particular period, and when the institution has to meet a periodic liability with a fixed floor, these draw downs can prove calamitous.

Attempting to profit from the virtually uninterrupted sequence of very good stock returns in every single year since 2002, pension fund managers have been steadily increasing their stock market exposure. This strategy started to attract a considerable number of critics (e.g. Ambachtsheer (1987))), stating that it does not conform both to the prudent investor principle of fiduciary fund management outlined above, and the regulatory framework. And indeed lately, and perhaps a little belatedly, pension funds have been intensely substituting "safer" bond investments for their equity asset allocations on a scale rarely observed. Is it possible, though, that this mass flight to safety might have been to their own detriment? Is this an example of a "too little, too late" reaction? These are the sort of questions that we hope to be able to answer after the completion of the proposed research project.

2.5 Existing Research on Optimal Multiperiod Investment

So far, we have seen that pension funds and life insurance investment companies have some interesting characteristics, which distinguish them from other long-term institutional investors and puts them at a disadvantage when the economy slows down. Usually, the sponsor of a defined-benefit plan is obligated to pay some minimum level of benefits, regardless of the pension fund's investment performance. The fact that the liabilities of these investment entities have a fixed minimum, which they have to meet in each time period, means that such institutions are exposed to shortfall risk (Bodie (1991)) and cannot survive falling asset values by simply holding on to their portfolios and waiting for stock prices to pick up again.

The latter has been a cause of serious problems lately, since both pension funds and insurance companies are dependent to a great extent on capital markets for their current income - the International Financial Services London (2008) estimate that insurers, for example, have historically derived around 15-20% of their revenue from investments. The current downturn has significantly reduced the income stream and forced contractual savings institutions to liquidate part of their portfolios in order to meet the shortfall. Additionally, the funds that were not impacted so badly, realized how bad the situation was and decided to take pre-emptive action by moving to a more conservative asset allocation, which was tantamount to initiating even more equity sales. The literature confirms that such mass sales by large institutional investors can destabilize financial markets by means of both the sheer size of trades and their informational content, thus reinforcing the downward spiral of asset values.

So, the question then for pension funds is how to limit exposure to fluctuating investment income, so that they still have a comfortable reserve to meet their periodic payments? The answer seems simple - simply allocate all assets to fixed-income securities matching the duration of liabilities. This approach, however, will cause a fund to miss on the upside potential in good times and it might even fail to generate enough returns to surpass inflation rates and reach its investment target. Moreover, as Blake et al. (1999) note, the fact that fund management is a competitive industry, where managers are constantly evaluated against benchmarks and competitors, creates a strong disincentive against risking a large difference in relative performance. The solution, therefore, will be to find some optimum balance between risky, earning assets and cash equivalents or bonds to provide a cushion against adverse market moves. This objective drives the research project, and the three research questions delineated below are our proposed modeling framework to assist in our attempt to provide a solution to this problem.

Depending on one's perspective, the issue above can be formulated either as an insurance/risk management problem, an asset allocation problem, or an intertemporal problem of optimum investment-consumption. The following sections survey the most important papers in the literature directly related to the problem under study. These can be broadly classified under the headings of log-optimum investment strategies, utility maximization, and other more recent approaches, which either do not fall under any of these categories, or further extend them.

2.5.1 Log-Optimum Investment

2.5.1.1 Maximizing the Geometric Mean

The first major contribution in the analysis of how an individual should optimally invest over a long-term time horizon when he is faced with uncertainty during each time period is provided in Kelly (1956). In this article the author examines the problem of the optimal bet size if a person is repeatedly betting on the outcome of an event transmitted over a noisy channel. The reason why precisely the bet size is examined is because even in positive expected value games a person can go bankrupt if he ends up overbetting. For example, consider a situation where a gambler is playing a game of chance in which a fair coin is tossed. The gambler decides to bet a dollar: he wins two dollars if the result of the coin flip is heads or loses his stake otherwise. The objective is to complete the game with as large an amount of terminal wealth as possible after an arbitrary number of coin tosses.

Probabilistically this is a very profitable game since it has a positive expected value of fifty cents for every dollar wagered. Even in this situation, however, the gambler is not guaranteed to profit. Since the expected value of each bet is positive, by analogy with the single-flip case, a na?ve "optimal" strategy would be to bet one's entire capital for each bet: since with a single bet the game terminates after one coin flip, and since the game has a positive expected value, to reach the objective of a maximum terminal wealth, the player should participate in the game and venture all his available wealth.

This one-game strategy is sub-optimal in the long-run, however, since if the gambler decides to risk all of his capital on any one particular flip, then there is a 50% chance he will go bankrupt. As the number of bets gets large, the probability of eventual bankruptcy approaches unity (Thorp (2006)). Therefore, both maximum expected payoff and the game's expected value are poor criteria, on which to base successive decisions about risky events.

Conversely, and more importantly in light of our main question, if the gambler (or investor) wishes to minimize the probability of bankruptcy, the "optimal" strategy is to risk no more than the minimum required bet size (which would correspond to a pension fund having most or all of its assets in bonds and cash equivalents). The problem with this, as mentioned above, is that the hypothetical investor would also minimize their expected return with this approach. Therefore, a middle ground has to be established: for the gambler this means that some fraction of his bankroll should be wagered at each bet; for our pension funds - that there is some optimal asset allocation between risky and riskless assets, which would minimize the chance of bankruptcy without completely sacrificing returns.

Instead of expected value, the solution to this problem, proposed by Kelly (1956), bases the decision on expected utility. The utility function of choice is the logarithmic one. So, in essence the Kelly criterion seeks to maximize the expected value of the logarithm of wealth, which in financial terms would mean maximizing the rate of growth of one's wealth: it is not the level of wealth or expected payoff that is important - it is the rate at which wealth grows over time. Since the Kelly criterion supposes that earnings are reinvested each period, it is also known under names such as "geometric mean maximizing strategy", alluding to the compounding of wealth that takes place during reinvestment.

Skipping most of the details, the expected value of the exponential rate of wealth increase per period in n periods is given by:

$$g(f) = E\left[ln\left(\frac{X_n}{X_0}\right)^{\frac{1}{n}}\right] = qln(1+bf) + (1-q)\,ln(1-f),$$

where X_n is the wealth accumulated by period n, q is the probability of winning, b is the payout for each wagered unit in case of a win, and f is the proportion of wealth to be bet each time. To maximize the expected value of this growth coefficient, first-order conditions are used. Differentiating the above expression with respect to and setting the derivative equal to zero yields:

$$f^* = \frac{q(b+1)-1}{b}.$$

In other words, the optimal bet size is the difference between the probabilities of winning and losing adjusted for the possibility of uneven wager payouts. This proposed formula for the percentage of wealth one should bet (or invest) in each period guarantees that wealth would grow at its quickest if the gambler is playing a positive expected value game. Thorp (2006) proves mathematically that an investor following the Kelly criterion in an infinite (or very long) series of bets will outperform any other investor following a different strategy with probability equal to one. By betting the fraction proposed above, the player is at all times guaranteed that he will not be completely wiped out by a longer losing streak. A very important feature of the Kelly criterion is that unlike expected value, the order of winning and losing bets does not matter.

Kelly's criterion is also favoured by Latané (1959) in his proposed structured approach to decision making under uncertainty, which includes defining a goal, a subgoal, and a criterion for choosing among strategies to reach the subgoal. The author deals with a very similar problem of investment decision making under uncertainty when the choice is repetitive and has cumulative effects. In this case the goal is defined as the maximization of wealth at the end of many periods. While the problem in this paper has no constraints concerning the minimum required wealth in each period, the differentiation of two possible subgoals is important in deciding which one is more appropriate to be adopted in our context.

The two subgoals examined by Latané (1959) are utility maximization and what he calls the P' subgoal, which is very similar to the Kelly criterion in requiring the choice of a strategy, which will lead to as much or more wealth as any other different strategy with a greater probability.

The author argues that while the proposed P' subgoal is less general, it is more operational than the expected utility criterion, because of the difficulty of constructing a payout matrix in terms of utiles, especially in the case of a firm or a group of people - something which is well known to fund managers who deal with a large group of diverse individuals. In this case, considering a separate utility function for each plan beneficiary would be impractical unless we restrict ourselves to a single representative rational agent - a topic which has been hotly debated lately. That's why, similar to Latané (1959), our only assumption is that any economic agent prefers more wealth to less, and we let the dynamics of wealth in the market be our guiding criterion. In other words, as Browne (1997) puts it, survival in the marketplace and growth of wealth are seen as intrinsic objective criteria that are independent of any specific utility function.

2.5.1.2 Fractional Kelly

The Kelly criterion has a number of good and bad properties, analyzed in Ziemba (2005), but unfortunately it does not solve our problem, since it is inappropriate for investors with less tolerance for short and intermediate term risk. This is so because the Kelly criterion tells what the optimal asset allocation over the long term is, but it does not eliminate the possibility of large drawdowns when one is faced with a longer losing streak. Our pension fund can be assured that it will not be completely wiped out, but it may still not be able to meet its periodic payments.

One possibility is to invest less in risky assets than what the Kelly criterion would suggest (a strategy known as "fractional Kelly") in an attempt to minimize the downside potential. Such strategies are examined in MacLean et al. ((1992), (2004)). The first of these examines how fractional Kelly strategies can be applied with two different objectives in mind: maximum growth and maximum security. This is accomplished by formulating three growth measures together with three corresponding security measures. The authors examine these pairs of measures by constructing an efficient frontier and then deriving from it a fractional Kelly path. Both curves exemplify the trade-off between growth and security. The authors also show numerical implementations for a number of gambling and investing situations.

An extension of this idea is presented in MacLean et al. (2004). The authors present a similar framework of dynamic allocation between stocks, bonds and cash equivalents. New measures are introduced - for example, for the security measures the emphasis is on value at risk and controlling period-by-period drawdown. Again a log-utility criterion is used, but this time two regions are considered: a risky one and a safe one. The model is implemented via a numerical simulation and the results suggest that, unsurprisingly, at low levels of risk control the Kelly criterion is still optimal. As the risk requirements are increased, the strategy turns more conservative with the corresponding reduction in return. The somewhat surprising result is that fractional Kelly strategies are generally not optimal at higher levels of risk control.

2.5.2 Utility Maximization

2.5.2.1 Multiperiod Portfolio Choice and Asset Allocation

The log-optimum approach presented above is not without its critics. Some of the more mainstream authors, whose work can be classified under the current heading (e.g. Samuelson (1969) and Samuelson (1971)), even classify it as a "fallacy". The common feature of the papers presented below is that they all consider the multiperiod investment problem in terms of utility and risk aversion functions, which they usually solve by means of dynamic stochastic programming, either in discrete or continuous time.

One of the two important papers that set the stage for this kind of analysis was Mossin (1968), who extended the one-period portfolio choice problem, often discussed in modern portfolio theory, to a multiperiod setting. The author's main critique to previous attempts at solving this problem is the focus on portfolio rate of return, rather than on the absolute size of the portfolio. This is an important consideration for large players in the market like pension funds.

The model framework for most of the papers in this thread of research is the following. An investor is faced with an intertemporal investment problem: this might be either how to optimally split wealth between investment and consumption, or how to split it among different asset classes. The investor makes a decision at the beginning of each period, but due to the uncertain nature of stock returns he has to wait till the end of the period, when the return on his portfolio materializes. He then has to make a decision for the next period. The investor cannot make any intermediate changes to his portfolio composition. The objective is to maximize the expected utility of wealth at the end of the final period. Generally, two common risk aversion measures are employed: absolute risk aversion and relative risk aversion.

In the simplest case outlined in Mossin (1968), there are only two assets - a risky one paying a random rate of return of X units per each dollar invested and a riskless asset with a certain rate of return of zero (a cash equivalent). Subsequent papers extend this to the case where there is also a fixed income instrument paying a known rate of return but this does not significantly alter the results. For our research project we also start with a case similar to the above, the only difference being that we use two different risky assets. The overriding conclusion from this type of models is that the investor will hold a positive amount of the risky asset if and only if its expected return is positive.

Mossin (1968) reconciles his analysis with previous work in the field and derives a utility function consistent with the result that investment in the risky asset is strictly proportional to wealth. Additionally, a utility function that allows for myopic decisions is also presented. This author also provides a more general model specification of the multiperiod investment problem, in which consumption takes place only at the end of the final period when terminal wealth is known.

This decision of separating investment and consumption choices is understandable in light of the existing analytical tools available to researchers in this period, but is not very realistic since the two decisions are related. The author himself acknowledges this shortcoming. Other deficiencies include the fact that the proposed model cannot be trivially extended if any statistical dependence is present (e.g. serial correlation in returns), as well as the haphazard nature of the solutions - one could think of and try different utility function specifications, which would materially affect the results. These shortcomings, combined with the fact that utilities are unobservable in practice and are not suitable for decision making when groups of individuals are involved, decrease the ability of this class of models to provide a unequivocal answer as to what wealth splitting strategy will ensure the survival of pension fund in turbulent times and how this will impact on asset price dynamics.

This is so due to the characteristic feature of fixed periodic payout liabilities of contractual savings institutions, which we treat as consumption (or outflows) in our model. Generally, extant literature in this category deals either with the question of how to optimally invest one's wealth among two or more assets over time disregarding consumption, or how to choose between investment (usually in one asset only) and consumption in a multiperiod setting. As was demonstrated above, introducing a consumption constraint in the asset allocation problem is an important issue for the survival of pension funds and our main contribution aims at marrying the asset allocation and consumption decisions together in a unified framework.

2.5.2.2 Optimal Intertemporal Investment-Consumption Decision

Apart from Mossin's (1968) early work on portfolio selection and asset allocation, the other theme - optimal investment-consumption decisions in a multiperiod setting - has also been fairly well examined. Starting with Ando & Modigliani (1957), the life cycle model of saving and consumption has become a cornerstone paradigm in economic theory. It was subsequently extended and generalized on numerous occasions - for example in Merton (1969) and Merton (1971), as well as the overlapping generations model in Samuelson (1958) and Diamond (1965).

The life cycle model concerns itself with the problem of an individual facing a decision as to how he should finance his consumption during the years of retirement. The main motivation is to accumulate enough assets so as to support habitual consumption in retirement. The model is also motivated by the empirical observation that the consumption of a representative individual is usually much smoother through time than their income. This creates a mismatch between assets and liabilities in the form of a minimum required subsistence consumption level.

In the life cycle model, an individual's life is divided in two periods: youth and retirement. If an individual does not reduce their consumption below the income level during the working stage and accumulate some reserve assets, they will experience a fall in living standards later. This modelling framework treats consumption as a function of total wealth, which can be financed either out of current income, or the sale of accumulated financial assets. The life cycle approach readily accommodates the two-fund separation theorem by allowing borrowing in one's earlier years and lending in later years, thus separating funding decisions from the individual's objective: maximizing the lifetime utility of wealth.

Again, the approach here is similar to the one in Mossin (1968), but is closer to our problem since it allows consumption in all periods and not just in the final period. The individual's life is partitioned in a large number of successive periods, where the individual maximizes the discounted value of utility from consumption in all periods, subject to an intertemporal budget constraint.

A crucial concept in the life cycle model framework is the intertemporal substitution elasticity, also known as the rate of time preference, which measures the individual's preference for current consumption against future consumption. In order for a solution to be derived, one must once again assume a specific utility function. Early papers in this field tend to use a logarithmic function, similar to Kelly (1956), but later papers extend the analysis to a broad class of utility functions.

Despite the life cycle model's reliance on utilities, and while it is generally unconfirmed by empirical evidence (Blake (2006)), it still represents a useful framework to use as a starting point for our problem. In the case of pension funds and insurance companies, the choice to make is whether to invest the greater portion of funds in risky assets in the hope of maximizing expected return and market share, or to allocate some of them to more conservative asset classes such as bonds or cash.

2.5.2.3 Critique to the Log-Optimum Investment Argument

The early work on log-optimum investment and the Kelly criterion, presented in the section on log-optimum investment, was subsequently criticized by proponents of the utility maximization approach (see e.g. Samuelson (1969), Samuelson (1971), Merton (1969), Merton & Samuelson (1973)). The main argument is improper interpretation of the asymptotic properties of the Kelly criterion.

Samuelson (1971) argues that although the law of large numbers and the central limit theorem, when applied to logarithms, can validate the asymptotic property that a maximum-geometric-mean strategy will eventually maximize terminal wealth and utility, this does not imply the interpretation that such a strategy is optimal for any finite number of periods, no matter how long, or that it becomes asymptotically a good approximation. To confirm this statement, the author provides an interesting counter-example - the case of a utility function of the form:

$$U(X) = \frac{X^y}{y}, \quad y \neq 0.$$

In fact the author's analysis shows that for any utility function bounded from above and finite at zero wealth, no uniform strategy can be optimal. The non-uniform strategies, which are optimal in such a case, are less aggressive than the Kelly criterion, in the sense that they would recommend a larger allocation to risky assets at low wealths and a smaller risky allocation at high wealths.

Having exposed these caveats with regard to the log-optimum investment approach, Samuelson (1969) and Merton (1969), show their own perspective on the problem of intertemporal consumption-investment decisions. In two independent, but related papers, the authors analyze the problem of optimal lifetime portfolio selection in discrete and continuous time respectively. Of particular interest is the concept of the so-called "businessman risk", according to which conventional wisdom would recommend that when faced with an investment decision regarding a risky asset, an individual with a shorter investment horizon, such as a widow for instance, should avoid the asset. On the other hand, however, the risky asset should be acceptable for a successful businessman, who has a long-time horizon, and who can expect earnings in later stages of his life to be large enough to offset any possible losses from this particular investment.

The main ambition of both of the aforementioned papers is to establish the plausibility of claims that the averaging effect of a long time horizon, combined with a higher expected value of future income, is enough to change the standard mean-variance optimizing behaviour an individual would display in a single-period setting, by causing them to exhibit a greater risk preference. In the end, both Samuelson (1969) and Merton (1969) deny the "businessman risk" argument, by showing that under the assumption of isoelastic marginal utility, a person with a long time horizon would have the same relative risk-tolerance as an individual with a shorter investment span. Using this and other results, the authors provide another justification for disagreeing with the Kelly criterion. They argue that, in general, even when a certain investment rule almost surely dominates other strategies in terms of their long-run growth rates, it is erroneous to assume that it also automatically yields a higher expected utility.

The reason provided for this conclusion is that any decision rule of selecting between two choices, on the basis of which has the greater probability of producing a higher result, does not possess the property of being transitive, which is a key property in the definition of a preference relation. Samuelson (1969) and Merton (1969) also show that for the family of constant relative risk aversion utility functions, the length of the investment horizon per se has no predictable effect on the proportion to be allocated to risky assets. This result disproves the widely-spread misconception that because stocks have outperformed both bonds and inflation over the long run, and since pension funds are long-term investors, it is best to have most, if not all, of the fund's assets in equities. The analysis started in 1969 was then further extended in Merton (1971) by including more general utility function specifications and price behaviour assumptions. These generalizations ultimately culminated in the creation of the intertemporal capital asset pricing model (ICAPM), which unfortunately suffers from the same shortcomings as its one-period predecessor (Merton & Samuelson (1973)).

2.5.2.4 The Term Structure of Risk: Investment Horizon vs. Holding Period

This distinction between the peculiarities of short versus long-term investing over many periods became a popular theme after the introduction of the ICAPM and many contributions followed. In an influential paper, Fischer (1983) starts his analysis from the empirical observation that expected real returns on Treasury bills are serially correlated and thus have different dynamics as compared to stocks. The relative risk of these two asset classes is characterized not only by the volatility of their returns, but also by the length of their holding period.

Stocks are of course much riskier than bonds when viewed over a short-term horizon. But does this hold true for the long-run as well, bearing in mind the serial correlation in returns? Fischer (1983) examines three questions to help pension funds and other long-term institutional investors with their asset allocation decisions: first, how does the term structure of risk arising from differences in the dynamics of asset returns affect optimal long-term investment behaviour; second, an empirical study on the dynamics of return on stocks and bills in the United States; and third, given the statistical estimates derived in the second part, how does the optimal portfolio for a long-term institutional investor change with the length of the holding period?

In Fischer's (1983) model the investor maximizes an intertemporal utility function. Of particular interest to our problem is the distinction that the author makes between the notions of "investment horizon" and "holding period". The former is closely related with the specific investment objective.

The papers discussed in the previous few paragraphs deal solely with the effect of the length of the investor's horizon on optimal portfolio composition.

The first main question was whether the investor should strive to maximize the expected growth rate of the value of the portfolio, as suggested by Kelly (1956). The answer turned out to be no - for utility functions with constant absolute or relative risk aversion, the investor's portfolio decisions are dependent only on wealth. The second question was will the lengthening of the horizon tend to change the behaviour of investors and as this lengthening takes place will all investors hold the same portfolio? Again the answer was no.

The investor's holding period, on the other hand, is defined as "the interval of time between successive portfolio actions". This is an important consideration, since while pension funds are long-term investors, their holding period might actually turn out to be short-term if rebalancing of the portfolio is needed in order to meet the fixed periodic liability. Such rebalancing would leave institutional investors exposed to the short-term volatility of asset prices. Fischer's (1983) paper focuses precisely on this point and rightly so, because Goldman (1979) shows that the composition of the optimal portfolio is not independent of the holding period, even when utility functions have constant risk aversion. The solution methodology is similar to Mossin (1968) and uses recursion relations.

The results suggest that the minimum variance portfolio moves toward or away from stocks as the holding period lengthens, depending only on the sign of the difference between serial correlations of returns in successive periods. Unlike the results obtained for the investment horizon, the portfolio composition in this model turns out to be highly sensitive to the length of the holding period. Also, diversification benefits from combining stocks and bonds only occur for utility functions with high risk aversion. A potential weakness of the study is that having obtained these theoretical results, the author tries to empirically estimate a data generating process for the returns of bills and stocks, and then uses the obtained parameters to calibrate a stochastic simulation. The results of the latter, however, turn out to be somewhat at odds with his theoretical findings, suggesting that optimal portfolios change little as the length of the holding period changes. Fischer (1983) explains this incongruity with the high degree of uncertainty inherent in empirically estimating data generating processes for stock and bond returns.

2.5.2.5 Portfolio Insurance

Both the log-optimum and the intertemporal utility maximizing approaches, discussed in this and the previous sections, were later integrated in a much more general and flexible framework - portfolio insurance. Black & Perold (1992) in their seminal paper define a portfolio insurance strategy as "any rule that takes less risk at lower wealth levels and more risk at higher wealth levels".

Such strategies are suitable for investors who want to have downside protection for their portfolios to insure against poor investment performance, while still maintaining their upside potential.

It is easy to see that the Kelly criterion is a subset of the more general class of portfolio insurance strategies. On the other hand, methodologically, the best portfolio insurance strategy can be found by solving for the intertemporal investment-consumption rules that maximize expected utility. As discussed above, this was accomplished in papers such as Merton (1971) and Leland (1980). The problem with these preceding papers is that their assumptions are too limiting - for example, it is common practice to assume frictionless markets and no borrowing restrictions. Introducing transaction costs and borrowing constraints changes the problem to a great extent, since it induces path dependency and other complications, which cannot be easily dealt with under such limiting assumptions.

In an attempt to bring analytical work closer to decision making in actual markets, Black & Perold (1992) dispense with the assumptions that were the generally accepted norm in earlier papers. The authors do not solve for the utility maximizing strategies, but rather take a stylized decision rule as a given and examine its properties. The portfolio insurance rule used in this particular paper invests a constant multiple of the cushion in risky assets up to the borrowing limit, with the cushion being the difference between current wealth and a prespecified floor. It turns out that this borrowing-constrained rule is utility maximizing as well, making the narrower intertemporal utility maximizing approach discussed above, a subset of this more general framework. In addition to the borrowing constraint, in essence intertemporal consumption is also constrained above the predetermined floor, which the authors refer to as "subsistence level".

The model specified in Black & Perold (1992) considers two assets: one is the so-called "safe" asset, and the other is a "risky" asset. The safe asset need not necessarily be a bond, but any instrument, whose income stream closely tracks the fixed liability. The risky asset fluctuates in value more and has a higher expected return. To achieve portfolio insurance, the investor trades actively, frequently rebalancing his portfolio. The number of trades is proportional to the number of prices reversals. This kind of rebalancing in times of reversals is costly because it involves buying high and selling low - not unlike the situation we are witnessing at the moment, with pension funds liquidating investments at unfavourable prices to keep up with their liabilities. Black & Perold (1992) refer to this occurrence as "volatility cost".

A larger multiple of available wealth invested in the risky asset means a larger market exposure, which leads to more frequent rebalancing trades in order to protect the downside. The results obtained in Black & Perold (1992) show that as the multiple goes to infinity, the constant proportion portfolio insurance converges to a stop-loss strategy, whereby as long as wealth is above the floor, the investor allocates everything up to the borrowing constraint to the risky asset, and then switches completely to the safe asset once wealth falls to the subsistence level. The authors also find that expected holding-period return is not monotonic in the multiple - higher returns can be achieved through ordinary constant proportion portfolio insurance strategies than through a stop-loss strategy. This comes as a surprise, since the latter is actually the most aggressive strategy in terms of risk exposure and one would have expected to see a commensurately high holding-period return. As a by-product of their analysis, Black & Perold (1992) find that the constant proportions strategy examined in their paper is actually equivalent to a perpetual American call option.

Unlike previous work, the model set-up in Black & Perold (1992) incorporates market frictions, such as transaction costs, borrowing constraints, and a fixed consumption floor. This additional richness brings their model much closer the problem we would like to analyze. Another strong point of the paper directly applicable to our study is the "reverse engineering" solution methodology, which first imposes a simple rule and then examines investment results as it changes, without solving for it explicitly using utility maximization. Such a maximization problem would be analytically intractable in this more flexible case of fewer assumptions. The trading strategies used in the model are not fixed and can easily accommodate dynamic asset allocation.

Modifications and extensions of certain aspects of the benchmark model first discussed in Black & Perold (1992) are frequently encountered in academic and professional journals in the field of financial economics. Dybvig (1995) extends previous work by examining a new class of portfolio and consumption strategies exhibiting ratcheting of consumption, as opposed to the constant proportions strategy discussed in Black & Perold (1992). This approach is motivated to an extent by the earlier work of Ingersoll (1992), who suggests that in the lifetime consumption-investment framework, the assumption that lifetime utility is additive may lead to an incorrect modelling of important dependencies in utility, such as intertemporal risk aversion and habit formation.

The additional constraint introduced in this model is the requirement that consumption can never fall - it remains constant most of the time and increases proportionately as wealth achieves a new maximum. After that it can never fall below this level, even if wealth diminishes. The justification for this consumption specification is an extreme version of habit-formation preferences - an individual gets accustomed to a certain level of consumption and experiences disutility in any consumption reductions. In fact, if consumption ever falls, utility is defined to be minus infinity.

The solution to the model exhibits a trade-off between the desire to smooth consumption against the desire to take advantage of expected profit opportunities in the stock market. This is a similar dilemma to the one faced by pension funds and insurance companies. Also, as discussed above, while there is a guaranteed minimum, "consumption" or periodic outflows in the case of contractual savings institutional investors are not fixed - the sponsoring company may decide to pay out more or less depending on a variety of factors, including investment performance of the plan portfolio, but it cannot go below the guaranteed minimum. So, a specification with a variable (but not necessarily ratcheting) consumption and a fixed floor seems appropriate for our problem.

Yet another consumption modification is provided in Browne (1997). The author studies the optimal behaviour of an investor, who is forced to withdraw funds (either to meet a liability or for consumption) continuously at a fixed rate per unit of time. The assets used for modelling are again risky stocks, whose dynamics follows a geometric Brownian motion, and riskless assets with a constant rate of return. Withdrawals are continuous and take place regardless of the wealth level, which means that there is a positive probability of ruin. The region where this is true is called "danger zone"; alternatively there is also a complementary region, where ruin can be avoided altogether with certainty - the "safe region".

Similar to MacLean et al. (1992), Browne (1997) focuses on survival and growth separately. The survival objective is to maximize the probability that the safe region is reached before bankruptcy. Once survival is ensured, attention is then focused on growth, or what investment policy will allow the investor to reach a prespecified goal as quickly as possible. The author finds that an optimal survival policy does not exist for this problem, but is able to construct a so-called "ε-optimal" strategy, which is within ε of optimality. The results pertaining to growth are similar to those in Black & Perold (1992).

2.5.3 Recent Contributions

A number of more recent papers explore problems related to pension funds or insurance companies in similar contexts to ours. These cannot be conveniently classified under one heading as the specifics of the problems analyzed, as well as the modelling methodology, differ to a considerable extent.

Early analytical work on pension funds and other institutional investors includes Antoci (1995), who develops a theoretical model of asset price and institutional wealth dynamics. Stocks and bonds are the only asset classes considered and the main motivation of the model is to examine the degree, to which institutional investors can affect the stability of financial markets through herding and positive feedback behaviour. Although optimal investment-consumption decisions are not studied here, the paper's modelling approach of how institutional trading affects the dynamics of asset prices certainly has value for the problem considered in our research project.

A model specification, which corresponds more closely to the issues we are facing, is studied in Hainaut & Devolder (2007). The authors consider the dividend policy and asset allocation decisions of pension funds under mortality and financial risks. Similar to the approach adopted in previous papers, the authors include three assets: a stock, cash and a bond. They then solve for the policies that maximize the utility of dividends and terminal surplus under a budget constraint by means of dynamic programming. In terms of utility function specifications, Hainaut & Devolder (2007) consider the cases of constant relative and absolute risk aversion. Unlike some more stylized papers, additional details are included: interest rates are driven by Vasicek's model, while the mortality of plan members is modelled by a Poisson process. Asset prices are exogenously modelled by a diffusion process, which means the possibility of price impact cannot be easily accommodated.

This gap in Hainaut & Devolder (2007) is tackled by a number of papers that focus specifically on liquidity risk and the closely related occurrence of price impact. Vath et al. (2007) examine a model for portfolio choice with one risk-free and one risky asset, subject to both liquidity risk and price impact. The standard approach in mathematical finance, which specifies asset price dynamics exogenously as a diffusion process, assumes perfect elasticity of traded assets - economic agents act as price takers, so they buy or sell arbitrary asset quantities without affecting their market price.

From the discussion above it is clear that large trades move the price of underlying assets, so such a specification is not appropriate when modelling the behaviour of institutional investors. Price impact and the accompanying illiquidity are important consideration in the short-term and are modelled by the authors. Vath et al. (2007) also recognise another deficiency in the standard approach. This prompts them to build their model in discrete time, since in reality investors face a number of restrictions, which do not allow them to rebalance trading strategies continuously. The trader's objective is once again to maximize the expected utility of terminal wealth over a finite horizon, subject to a solvency constraint. A solution is derived using an impulse control methodology.

Focusing more on liquidity risk for the case of insurance companies, Giorgi (2008) examines the impact of both the success of investment strategies and of liquidity shocks during unfavourable years on the wealth dynamics of insurance companies. The author is concerned more with long-term survival of insurance companies in cases where indemnities to be paid exceed collected premia, rather than optimal asset allocation. Even so, the author proposes an interesting modelling framework inspired by evolutionary reasoning, which provides a direct link between the successful investment performance in terms of growth of wealth and market share, and long-term survival. These are believed to be strongly related and are thus analyzed conjointly within a unified framework. This differs from the approach discussed above, which segregates the growth aspect from survival, and chooses different metrics to examine the performance of a certain strategy with respect to one or the other, but not both at the same time.

Giorgi (2008) notes the difficulties, which we discussed above, in applying the prudent investor rule in practice. He develops a no-bankruptcy condition for the investment and risk management strategies of insurance companies, which guarantees that in the presence of any type of competitor or trading strategy, institutional investors following the proposed investment approach will be able to face liquidity shocks almost surely.

The paper by Giorgi (2008) provides an interesting contribution over previous market selection literature (e.g. Blume & Easley (1992), Evstigneev et al. (2002), Hens & Schenk-Hoppé (2005), Evstigneev et al (2006)) in showing that when withdrawal (or "consumption") rates are non-proportional or negative, due to some additional constraint, such as liquidity shocks or in our case - a guaranteed minimum floor regardless of the wealth level, then in addition to the analysis of the performance of investment strategies, other factors need to be considered as well. The author shows that if this is not achieved, even a strategy with the maximal exponential growth rate can disappear from the market in the event of an exogenous liquidity shock. Interestingly, the minimum no-bankruptcy condition derived in Giorgi (2008) is more stringent than most statutory solvency requirements - in fact, it is shown that an insurance company that is content with simply satisfying regulatory constraints faces a strictly positive probability of bankruptcy and will eventually vanish from the market.

Giorgi (2008), however, is unable to provide a definitive answer with regard to exactly which investment strategy is evolutionarily superior in the case of liquidity shocks - the results suggest that given different assumptions for the dividend and liquidity shock processes, either there exists a unique evolutionary stable strategy, or no such strategy exists. The latter case is interesting because it shows a case when markets can remain evolutionarily unstable over a long period, which means that there is no single superior strategy that is able to increase its wealth faster and eventually drive its competitors out of the market.

The paper by Giorgi (2008) is important because it introduces the dimension of risk to the long-term survival literature. In his model, uncertainty relates mainly to the exogenous process generating the insurance claim. In our problem, investment success and survival are even more intimately related, since for the case of pension funds, the fixed minimum is specified and known in advance. The main risk comes from investment success and periods of asset price depressions.

Because of price impact and the endogenous asset price mechanism that we impose, however, the pension fund has a direct opportunity to influence asset prices and market share in its favour, and that's why good investment performance and asset allocation become even more important for long term survival. Thus, it can be stated that our main contribution lies in the addition of the dimension of short-term dynamic asset allocation, and its significance to long-term survival, to the market selection literature.

Other related pieces of work focusing on liquidity and the probability of ruin for insurance companies are Berry-Stolzle (2008) and Azcue & Muler (2009). The former considers the problem of optimal liquidation strategies and asset allocation decisions for the case of property insurance companies. Investors in Berry-Stolzle's (2008) model deal with both liquid and illiquid assets, which is somewhat similar to the problem considered in Giorgi (2008). The proposed approach is based on cash flows, however, and the greater share of the analysis is reserved for the question what the optimal selling strategies should be. Asset allocation is considered only as an initial asset allocation decision, which, in conjunction with an appropriate liquidation strategy, is found out to be capable of minimizing the capital required to cover claims for a predetermined ruin probability.

Similarly, minimizing the ruin probability for insurance companies in a model with borrowing constraints is the primary objective in Azcue & Muler (2009). The aim of the authors has some relevance to our problem, since they are considering a dynamic choice of investment policy for the surplus of an insurance company, which will minimize the ruin probability. The optimal value function is characterised as a solution to the Hamilton-Jacobi-Bellman equation, subject to a number of assumptions regarding the processes that drive the surplus of the insurance company and the dynamics of the risky asset.

A closely related paper that uses an almost identical model and solution methodology is Bai & Guo (2008), who study a model, in which insurers can once again invest in risky assets or buy proportional reinsurance. Two optimization problems are studied separately: maximizing the expected exponential utility of terminal wealth, and minimizing the probability of ruin. The solutions are derived from solving the corresponding Hamilton-Jacobi-Bellman equations. The results validate our earlier conclusion that investment performance and minimizing the probability of ruin are complementary objectives and should be studied together. Moreover, in Bai and Guo's (2008) solution, they even turn out to be equivalent for some special parameter choices.

Optimal proportional reinsurance and investment policies, but this time in a framework with transaction costs, are the main theme in Zhang et al. (2009) as well. The innovations in this paper are that the surplus of the insurance company, which is to be invested in risky assets, follows a linear diffusion process, and that the authors control for total risk by means of a conditional value-at-risk (CVaR) measure. Again, maximizing the expected exponential utility of terminal wealth is the main objective. The solution methodology is the same as in Bai & Guo (2008).

A slightly more general case, which is more closely related to our problem, and is not necessarily confined to the case of insurance companies, is presented in Detemple & Rindisbacher (2008). The authors consider a dynamic asset allocation problem in the presence of liabilities. Similarly to the above papers, however, the solution approach uses expected utility. The main innovation of the paper in this respect is the inclusion of a parameter the controls the tolerance for ending up with a shortfall in the funding ratio at the terminal date.

Unfortunately, the model specification in this specialized case does not consider market dynamics and is not capable of trivially incorporating the two-way link between asset allocation decisions and the resulting influence on asset prices. This nexus between investment strategies and market dynamics is of prime importance for our study, since the asset values resulting from the actively changing financial market dynamics feed into the fund's investment performance and have a direct influence on the success of the chosen asset allocation policy. It might very well turn out that the price impact of institutional investors and the concomitant change in asset price dynamics leads to the optimality of a different asset allocation strategy, and thus renders the solutions derived in this paper invalid.

Several other loosely related papers exist in this field. The topic and focus of these, however, gets progressively further away from the main questions that we hope to answer by undertaking the current research project (see e.g. Zhou et al. (2008)).

Chapter 3 A Model of Financial Market Dynamics

Contents

This chapter presents a stylized model of financial market dynamics. The author's point of departure is a general formulation of a simple financial market, influenced by the literature on evolutionary finance. The presentation of the basic model draws from expositions of evolutionary dynamics as laid out in Evstigneev et al. (2002), Evstigneev, Hens & Schenk-Hoppé (2006), and Evstigneev et al. (2008). Extensions of the general evolutionary finance model to the case of incomplete markets, as well as in a continuous-time setting are presented in Hens & Schenk-Hoppé (2005) and Palczewski & Schenk-Hoppé (2010b) respectively. A broad survey of related results is available in Evstigneev et al. (2009).

The aim of the aforementioned work is mainly to examine the results of asymptotic market dynamics, as well as to draw conclusions on the optimality of a certain class of long-term investment strategies and their related implications for asset pricing. Conversely, the main focus of this research is the comparison and viability of various investment strategies viewed from the perspective of an institutional investor with minimum guaranteed liabilities. This additional constraint, coupled with the stochastic nature of asset payoffs, means that an examination of asymptotic dynamics is not of particular interest in this context, since the random asset payoff process will almost surely cause investors to not be able to meet the guaranteed liability at some point in the long run.

Instead, we focus on a medium-term investment horizon and analyze the outcome of the market dynamics numerically. To this end, we relax a number of assumptions, frequently imposed in the literature on evolutionary finance in order to facilitate the derivation of analytic results. Additionally, we also borrow from the field of agent-based modeling in an attempt to capture the heterogeneity of investment strategies followed by institutional investors. Specifically, we extend the literature on evolutionary finance by introducing a risk-free asset that will enable an intertemporal transfer of wealth, explicitly modeling the uncertainty in the economy by specifying two different asset payoff generating processes, and focusing on a medium-term investment horizon. The assumption of time-invariant investment strategies is also relaxed. Furthermore, the self-financing property and the market clearing mechanism of the model presented below allow an extension of the agent-based literature. In our proposed model, investment strategies are no longer separated from consumption decisions and wealth endowments. Combined with the additional consumption constraint of minimum guaranteed liabilities, this means that it is possible for the agents populating the model to enter periods of underfunding and even bankruptcy. The problem of bankruptcy raises some interesting questions not studied in detail in the agent-based literature thus far.

This chapter will first describe a general model of financial market dynamics. Additional extensions to this basic case will then be proposed. These comprise the inclusion of an additional consumption constraint in the form of a minimum guaranteed liability, two alternative specifications of a dividend generating process describing the uncertainty in the economy, as well as a discussion on the different time-varying investment strategies to be used in the model. These go beyond the simple fundamental and trend following strategies prevalent in other agent-based models and require the estimation of fundamental values for traded assets. To this end, agents attempt to estimate the unobserved states of the industries, in which the companies to whom the risky assets belong operate. This is accomplished by means of using observed realizations of the dividend processes and applying Bayesian updating to the conditional regime probabilities. Finally, details on the implementation of the model in the form of a computer program will be provided.

3.1 The Model

Our starting point is the formulation of a general model describing a stylized financial market. We choose to adopt a simple setup frequently used in evolutionary finance and mathematical economics.

Its continuous-time equivalent coincides with the description of the value dynamics of self-financing portfolios. This particular modeling approach is widely used in the field of mathematical finance (see e.g. Björk (2009), Shreve (2004)).

The reasons for this choice are several. Firstly, there are very few assumptions made regarding investor behaviour. A descriptive view of investors' actions is taken. The strategies that agents use are taken as model primitives and are not obtained from the general equilibrium setting of utility maximization. This leaves ample opportunity for the modeling of heterogeneous strategies capturing the most popular styles of investment that institutional investors tend to favour. Secondly, this approach relies heavily on the market interaction of investors. Market clearing in each period is ensured by equating supply and demand by means of an endogenous asset price formation mechanism. The latter allows the market impact of transactions to be implicitly considered for an investor with an arbitrarily large wealth endowment and not just for the case of a large trader. Price impact is an area of keen interest and intense research for large institutional investors, and hence needs to be considered. Our approach takes this into consideration but also exhibits a marked difference from other models that feature market impact, such as e.g. Bank & Baum (2004) or He & Mamaysky (2005), who use an explicit exogenous price impact function for the large traders.

3.1.1 Base case

The basic model of wealth dynamics presented in this chapter will be formulated in discrete-time. To achieve scalability with respect to time, we adopt the approach in Palczewski & Schenk-Hoppé (2010a) and define a dividend process in continuous time. The continuous-time specification of the dividend generating process allows these periods to not necessarily have a uniform length across time. The validity of such an approach, as well as its convergence to a continuous-time limit, which is well known in mathematical finance, is corroborated by existing literature (Palczewski and Schenk-Hoppé, 2010a). Such a formulation enables agents to adopt a custom trading frequency corresponding to their unique investment styles, regardless of the frequency and specification of any exogenous process providing them with information, on which they can base their decisions.

Dividend payments are derived from continuous-time intensities. Integrating the latter between the discrete time points of the wealth dynamics can be interpreted as an accumulation of dividends and payment of the sum at the end of each period in the dynamics. Note that these time periods may be of arbitrary length. In contrast to the aforementioned paper, however, the specific investment strategies studied in this research will be directly linked to the evolution of wealth. Trading only takes place at the end of each period – a procedure similar to the way institutional investors rebalance their portfolios at fixed time intervals with reference to a desired benchmark. Thus, investment strategies will also be formulated in discrete time.

Consider the following description of a financial market. Time is discreet and proceeds sequentially through the set $\{t_0, t_1, t_2, \ldots\}$, with $t_n < t_{n+1}$. There are K risky, long-lived assets (e.g. stocks), as well as a risk-free asset, which returns an interest rate r. Without loss of generality, the net supply of each risky asset can be set to one. The risk-free asset may be interpreted to be a liquid government bond. The price of the risk-free asset is fixed to one, and we will refer to it as a money market account. This instrument serves two important functions in our model. Firstly, it provides a means of intertemporal transfer of wealth. In previous evolutionary finance models, agents were constrained to two possible actions – reinvestment of the received dividend payments or their immediate consumption. This leaves no opportunity to accumulate a reserve without subjecting oneself to the risk of the stock market. Thus, the second function of the risk-free asset is to provide a means for risk management. Each of the K risky assets generate random dividend intensities $\delta(t) = (\delta_1(t), \ldots, \delta_K(t))$, $t \geq 0$, with $\delta_k(t) \geq 0$ for $k = 1, \ldots, K$. The precise specification of the intensities will be clarified later in this section. Since these intensities are stochastic, they depend on a random event $w \in \Omega$. Asset payoffs are aggregated into lump-sum payments paid to shareholders at the end of each time period. The total dividend paid by asset k at time t_{n+1} to the investors who hold the asset over the period $[\,t_n, t_{n+1}\,)$ is denoted by $D_{k,t_{n+1}}$, where

$$D_{k,t_{n+1}} = \int_{t_n}^{t_{n+1}} \delta_k(s)\, ds. \tag{3.1}$$

These dividends are paid in terms of a perishable consumption good, whose price is normalized to one. The prices of the risky assets are in fact relative prices that are expressed in terms of the consumption good price. As discussed above, since the available consumption in the present cannot be stored for the future, the consumption good is not a substitute for money.

The financial market is populated by I investor types with initial wealth endowments $V_0^i \geq 0,\ i = 1, \dots, I$. The total initial wealth in the marketplace, $\bar{V}_0$, is defined as the sum of the individual initial wealth endowments, i.e. $\bar{V}_0 = \sum_{i=1}^{I} V_0^i > 0$. Each investor type is represented by a vector of proportions $\lambda_{t_n}^i = \left(\lambda_{1,t_n}^i, \dots, \lambda_{K,t_n}^i\right), n \geq 0, 0 < \lambda_{k,t_n}^i \leq 1$. For each asset, λ_{k,t_n}^i describes investor i's budget share invested in risky asset k at the beginning of the period $[\, t_n, t_{n+1}\,)$. These proportions shall be hereafter referred to as investment strategies. Investment strategies depend on information available up to time t_n. Investors rebalance their portfolios according to their strategy at discrete points in time $\{t_0, t_1, \dots\}$. For the time being, trading strategies will be assumed to be given and possibly random. The author's choice of specific strategies and the computational procedure used to obtain them will be disclosed below.

Apart from the risky asset class, agents also have the option of investing in risk-free money market assets. The proportion of agent i's wealth invested in the money market at the beginning of the period $[\, t_n, t_{n+1}\,)$ is denoted by λ_{0,t_n}^i. Each investor consumes a certain proportion of their wealth in each time period. This consumption takes place at the end of the period $[\, t_n, t_{n+1}\,)$ and its amount is denoted by $C_{t_{n+1}}^i$. Consumption depends on information available at time t_n. In contrast to earlier work in evolutionary finance and agent-based modeling, the model outlined in this document combines investment and consumption decisions, each impinging on the other. Rather than providing a level playing field for all agents by assuming a uniform consumption rate, as argued in Palczewski & Schenk-Hoppé (2010b), investors will be penalized when an underfunding situation occurs by being forced to meet the minimum guaranteed liability regardless of their level of wealth. This leads to two different intensities at which agents consume depending on the size of their wealth endowment. Thus, we relax the assumption of common consumption.

For the purposes of simplification, we impose two assumptions on the investment behaviour of economic agents: budget exhaustion and full diversification. In each period all investors must use their available wealth in its entirety by investing in risky assets, consuming and saving any excess by investing it in the money market. Full diversification implies that agents have to split their wealth in such a way as to ensure a strictly positive allocation to each of the available risky assets, as well as to the money market. In other words, $\lambda_{k,t_n}^i > 0$ for all $k = 1, \dots, K$, and $\lambda_{0,t_n}^i > 0$ for all investors. Combining the assumptions of full diversification and budget exhaustion means that the condition $\sum_{k=1}^{K} \lambda_{k,t_n}^i = 1 - \lambda_{0,t_n}^i$ must be satisfied at the beginning of each period.

Rather than describing the investors' holdings of risky assets as proportions of their total wealth, it is possible to write an alternative formulation of the agents' asset allocation in terms of physical units – i.e. the number of shares that they own. Denoting the wealth endowment of investor i at time t_n by $V^i_{t_n}$, the investors' portfolios Θ^i_{k,t_n} at the beginning of the period $[\, t_n, t_{n+1}\,)$ are given by

$$\Theta^i_{k,t_n} = \frac{\lambda^i_{k,t_n} V^i_{t_n}}{S_{k,t_n}}, \quad k = 1, \dots, K. \tag{3.2}$$

The market for each risky asset would be in equilibrium if the demand from all investors equals supply. Since a positive net supply of one was assumed, this means that the total number of shares of asset k owned by all investors must be equal to one: $\sum_{i=1}^{I} \Theta^i_{k,t_n} = 1$. Consequently, the market for asset k at time t_n clears if its price S_{k,t_n} is set to

$$S_{k,t_n} = \langle \lambda_{k,t_n}, V_{t_n} \rangle, \tag{3.3}$$

where $\langle x, y \rangle = \sum_{i=1}^{I} x^i y^i$ denotes the scalar product. The advantages this market clearing mechanism provides are significant. Instead of assuming that all investors agree on the moments of the return distribution of each asset, which is a standard assumption in the context of mean-variance optimization, our proposed model relies merely on a short-term equilibrium, guaranteed by market clearing at the current point in time. Equation (3.3) explicitly measures the market impact of trades by the individual investors. The magnitude of the price impact caused by each agent is determined by both the size of their wealth endowment and their allocations to each risky asset. The asset price is simply the aggregate investment in each risky asset.

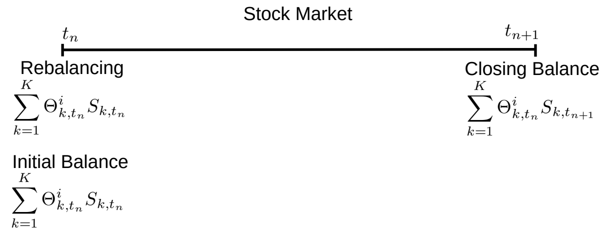
Randomness in our proposed model is caused by two sources – the agents' investment strategies and the variable dividend payments, which can be interpreted as a stochastic asset payoff process. This randomness serves as a driving force for the evolution of wealth. This approach represents a departure from the standard framework in mathematical finance, which specifies an exogenous stochastic asset price process and subsequently determines investors' portfolio allocations.

Similarly, the money market account can be interpreted as another asset with a constant price set to one. The notable difference here is that we impose no explicit market clearing for the risk-free asset. Later it will be shown that a condition that has to be satisfied by aggregate investor wealth provides logical market clearing levels of demand and supply for the risk-free asset and the consumption good.

Substituting for (3.3) in (3.2) shows that the portfolios can be rewritten as

$$\Theta^i_{k,t_n} = \frac{\lambda^i_{k,t_n} V^i_{t_n}}{\langle \lambda_{k,t_n}, V_{t_n} \rangle}, \quad k = 1, \dots, K. \tag{3.4}$$

The total value of investors' positions in the two markets, which result from the transactions they undertake in each of them during every time period, can be schematically summarized as follows:



Investor i's allocation to the money market account at the beginning of each period is $\lambda^i_{0,t_n} V^i_{t_n}$. Concurrently, investors also rebalance their risky asset holdings in accordance with their investment strategy. A net purchase of the amount $\sum_{k=1}^{K} \Theta^i_{k,t_n} S_{k,t_n}$ of securities is financed with funds from the money market account – the investors' insurance and wealth transfer mechanism. In the case of a net sale of risky assets, the proceeds are added to the money market account. At the end of the period, the funds initially invested in the money market will have grown by the prevailing rate of interest in the market to $\lambda^i_{0,t_n} V^i_{t_n} (1+r)$. Investor i receives a total dividend payment from their investments amounting to $\sum_{k=1}^{K} \Theta^i_{k,t_n} D_{k,t_{n+1}}$ and consumes $C^i_{t_{n+1}}$. Depending on the market price $S_{k,t_{n+1}}$ of risky asset k at time t_{n+1}, the total value of an investor's stock market position at the end of the period is given by $\sum_{k=1}^{K} \Theta^i_{k,t_n} S_{k,t_{n+1}}$.

Combining the value of the positions in both markets, the stochastic evolution of investor i's wealth between two consecutive points in time can be expressed by the discrete-time dynamics:

$$V^i_{t_{n+1}} = \sum_{k=1}^{K} \frac{\lambda^i_{k,t_n} V^i_{t_n}}{\langle \lambda_{k,t_n}, V_{t_n} \rangle} \left(S_{k,t_{n+1}} + D_{k,t_{n+1}} \right) + \lambda^i_{0,t_n} V^i_{t_n} (1+r) - C^i_{t_{n+1}}.$$

Replacing the asset price $S_{k,t_{n+1}}$ with its definition in equation (3.3) yields:

$$\begin{aligned} V^i_{t_{n+1}} &= \sum_{k=1}^{K} \frac{\lambda^i_{k,t_n} V^i_{t_n}}{\langle \lambda_{k,t_n}, V_{t_n} \rangle} \left(\langle \lambda_{k,t_{n+1}}, V_{t_{n+1}} \rangle + D_{k,t_{n+1}} \right) + \lambda^i_{0,t_n} V^i_{t_n} (1+r) \\ &- C^i_{t_{n+1}}. \end{aligned} \tag{3.5}$$

In other words, since agents consume and invest in a self-financing way, changes in the value of their wealth endowments can be attributed to either changes in asset values, the dividend income they receive, the value of their accumulated money market reserve, or their consumption. Alternatively, using vector notation:

$$V_{t_{n+1}} = \Theta\left(\Lambda_{t_n}, V_{t_n}\right)\left(\Lambda_{t_{n+1}} V_{t_{n+1}} + D_{t_{n+1}}\right) + M_{t_{n+1}} - C\left(V_{t_n}\right), \qquad (3.6)$$

where the matrix of investment strategies $\Lambda_{t_n} \in \mathbb{R}^{K\times I}$ is given by $\Lambda_{ki} = \lambda_k^i$, and the portfolios matrix $\Theta\left(\Lambda_{t_n}, V_{t_n}\right) \in \mathbb{R}^{I\times K}$ is defined as $\Theta_{ik}\left(\Lambda, V\right) = \frac{\Lambda_{ki} V^i}{(\Lambda V)_k}$. Vector $M_{t_{n+1}} \in \mathbb{R}^{I\times 1}$ denotes the value of the money market position at time t_{n+1} for each investor: $M_{i1,t_{n+1}} = \lambda_{0,t_n}^i V_{t_n}^i \left(1+r\right)$. Vectors $V_{t_{n+1}} = \left(V_{t_{n+1}}^1, \ldots, V_{t_{n+1}}^I\right)^T$, $D_{t_{n+1}} = \left(D_{1,t_{n+1}}, \ldots, D_{K,t_{n+1}}\right)^T$ and $C\left(V_{t_n}\right) = \left(C_{t_{n+1}}^1, \ldots, C_{t_{n+1}}^I\right)^T$ contain investors' total wealth, dividends received, and consumption respectively. For given investment strategies λ_{k,t_n}^i with the desired properties, (3.6) defines the dynamics for the vector of investors' wealth endowments. The necessary and sufficient conditions for this dynamical system to be well-defined will be given below.

The order by which trading takes place can be summarised as follows. At the beginning of each period t_n an investor has a certain amount of wealth invested in the money market account as well as a certain amount of wealth invested in the risky asset. At time t_n all investors make their decisions about what portfolios they wish to hold for the next period. Depending on total demand for the risky asset at time t_n, market clearing will ensure that the market is in equilibrium by setting a new price for risky asset k, denoted by S_{k,t_n}. The investors will then rebalance their portfolios transacting at that price. The asset price will then remain unchanged until time t_{n+1} is reached. At that point investors will make new asset allocation decisions and again set a new market clearing price $S_{k,t_{n+1}}$. Rebalancing can then take place at that price. From an accounting perspective, each period begins with portfolio rebalancing, right after the asset price has been set, and continues until a new asset price is set at the end of the period. After that a new period begins and rebalancing takes place again.

Apart from the fact that all components of the proposed model are observable, and can thus be empirically estimated, the formulation of the wealth dynamics above is also appealing from an economic point of view. The endogenous price formation mechanism in (3.3) ensures market clearing for the risky assets. The supply of the consumption good and the risk-free asset was left potentially unlimited in the discussion above. The self-financing property of the agents' investment strategies, combined with the assumptions of complete usage of investors' budgets and full diversification, suggests that a natural upper bound on the total supply of the consumption good and the risk-free asset would be the total amount of disposable income available to the agents after the desired investment in the risky assets has been completed. This feature of the model is in agreement with Walras' law from economics.

To see this, note that any solution to (3.6) possesses the following property of aggregate investor wealth. Summation of (3.5) over $i = 1, \dots, I$ gives:

$$\begin{aligned}\sum_{i=1}^{I} V_{t_{n+1}}^{i} &= \sum_{i=1}^{I}\sum_{k=1}^{K} \frac{\lambda_{k,t_n}^{i} V_{t_n}^{i}}{\sum_{i=1}^{I} \lambda_{k,t_n}^{i} V_{t_n}^{i}} \left(\sum_{i=1}^{I} \lambda_{k,t_{n+1}}^{i} V_{t_{n+1}}^{i} + D_{k,t_{n+1}}\right) \\ &+ \sum_{i=1}^{I} \lambda_{0,t_n}^{i} V_{t_n}^{i} - \sum_{i=1}^{I} C_{t_{n+1}}^{i}.\end{aligned}$$

Noting that the first term outside of the brackets on the left-hand side corresponds to the expression describing the investors' portfolios in equation (4.15), and using the property that $\sum_{i=1}^{I} \Theta_{k,t_n}^{i} = 1$ (net supply of one), the above can be rewritten as follows:

$$\begin{aligned}\sum_{i=1}^{I} V_{t_{n+1}}^{i} &= \sum_{i=1}^{I}\sum_{k=1}^{K} \lambda_{k,t_{n+1}}^{i} V_{t_{n+1}}^{i} + \sum_{k=1}^{K} D_{k,t_{n+1}} + \sum_{i=1}^{I} \lambda_{0,t_n}^{i} V_{t_n}^{i} \\ &- \sum_{i=1}^{I} C_{t_{n+1}}^{i}.\end{aligned}$$

Since $\sum_{k=1}^{K} \lambda_{k,t_n}^{i} = 1 - \lambda_{0,t_n}^{i}$, the above equation can be reformulated as:

$$\begin{aligned}\sum_{i=1}^{I} V_{t_{n+1}}^{i} &= \sum_{i=1}^{I} \left(1 - \lambda_{0,t_{n+1}}^{i}\right) V_{t_{n+1}}^{i} + \sum_{k=1}^{K} D_{k,t_{n+1}} + \sum_{i=1}^{I} \lambda_{0,t_n}^{i} V_{t_n}^{i} \\ &- \sum_{i=1}^{I} C_{t_{n+1}}^{i}.\end{aligned}$$

Simplifying the latter yields the final result:

$$\sum_{i=1}^{I} \lambda_{0,t_{n+1}}^{i} V_{t_{n+1}}^{i} - \sum_{i=1}^{I} \lambda_{0,t_n}^{i} V_{t_n}^{i} = \sum_{k=1}^{K} D_{k,t_{n+1}} - \sum_{i=1}^{I} C_{t_{n+1}}^{i}. \tag{3.7}$$

Equation (3.7) demonstrates the intertemporal wealth transfer and risk management properties of the risk-free asset. Unlike the much more stringent relations derived in Palczewski & Schenk-Hoppé (2010a) and Palczewski & Schenk-Hoppé (2010b) [equations (7) and (5) respectively], which imply that the condition $\sum_{k=1}^{K} D_{k,t_{n+1}} = \sum_{i=1}^{I} C_{t_{n+1}}^{i}$ must be satisfied in each period, the inclusion of a risk-free asset class allows greater flexibility. Consumption can now temporarily exceed current income, as long as investors have accumulated enough funds in their money market account. If investors wish to consume more than what is left from their total dividend income after they have invested in risky assets, they would have to rely on the their reserves to cover the shortfall. That is, $\lambda^i_{0,t_{n+1}} V^i_{t_{n+1}} < \lambda^i_{0,t_n} V^i_{t_n}$, or a negative change in the value of their savings.

Since short positions in either the risky assets or the risk-free asset are not allowed, when an agent's money market account contains insufficient funds to meet the required consumption, the investor would be forced to liquidate a part of their risky asset holdings in order to accommodate the excess consumption. In essence, the lack of short positions in the risk-free asset is equivalent to an inability to borrow. This addresses a common shortcoming in other agent-based models, where agents usually make their investment decisions based on expected utility maximization and are allowed to freely follow their investment policies without any regard to the size of their wealth endowment or required consumption. This situation is tantamount to letting investors have an infinite line of credit. Clearly, this drawback must be addressed if the goal is to realistically model the asset allocation challenges faced by pension funds and other institutional investors with minimum required liabilities.

Equation (3.5) is an implicit description of the wealth dynamics of investor i. Since the asset price at time t_{n+1} is specified in such a way as to ensure market clearing for the risky assets during this period, it directly depends on the wealth endowments of all agents during the period that we are trying to solve for and $V^i_{t_{n+1}}$ appears on both sides of the equation. This intertemporal problem can be overcome by using changes in wealth rather than the absolute wealth value in the formulation of the dynamics.

We begin by noting that at the beginning of each time period at time t_n, right after dividends for the previous period have been paid and all necessary consumption has taken place, the wealth of each investor can be represented as the sum of their stock market and money market positions: $V_{t_n} = \Theta\left(\Lambda_{t_n}, V_{t_n}\right) \Lambda_{t_n} V_{t_n} + M_{t_n}$, where $M_{t_n} \in \mathbb{R}^I$ denotes the vector of agents' money market investments at time t_n: $M_{i,t_n} = \lambda^i_{0,t_n} V^i_{t_n}$.

Expressing the wealth dynamics (3.6) in terms of the change in wealth between the beginning and the end of the period $[\, t_n, t_{n+1}\,)$, we have:

$$\begin{aligned} V_{t_{n+1}} - V_{t_n} &= \Theta\left(\Lambda_{t_n}, V_{t_n}\right)\left[\left(\Lambda_{t_{n+1}} V_{t_{n+1}} - \Lambda_{t_n} V_{t_n}\right) + D_{t_{n+1}}\right] \\ &+ \left(M_{t_{n+1}} - M_{t_n}\right) - C\left(V_{t_n}\right). \end{aligned}$$

Multiplying out the brackets on the right-hand side gives:

$$\begin{aligned} V_{t_{n+1}} - V_{t_n} &= \Theta\left(\Lambda_{t_n}, V_{t_n}\right) \Lambda_{t_{n+1}} V_{t_{n+1}} - \Theta\left(\Lambda_{t_n}, V_{t_n}\right) \Lambda_{t_n} V_{t_n} \\ &+ \Theta\left(\Lambda_{t_n}, V_{t_n}\right) D_{t_{n+1}} + \left(M_{t_{n+1}} - M_{t_n}\right) - C\left(V_{t_n}\right). \end{aligned}$$

Rearranging this, we have:

$$\begin{aligned} V_{t_{n+1}} - \Theta\left(\Lambda_{t_n}, V_{t_n}\right)\Lambda_{t_{n+1}} V_{t_{n+1}} &- V_{t_n} + \left[\Theta\left(\Lambda_{t_n}, V_{t_n}\right)\Lambda_{t_n} V_{t_n} + M_{t_n}\right] \\ &= M_{t_{n+1}} + \Theta\left(\Lambda_{t_n}, V_{t_n}\right) D_{t_{n+1}} - C\left(V_{t_n}\right). \end{aligned}$$

Using the fact that $V_{t_n} = \Theta\left(\Lambda_{t_n}, V_{t_n}\right)\Lambda_{t_n} V_{t_n} + M_{t_n}$, the equation above is equivalent to:

$$\begin{aligned} V_{t_{n+1}} - \Theta\left(\Lambda_{t_n}, V_{t_n}\right)\Lambda_{t_{n+1}} V_{t_{n+1}} &- V_{t_n} + V_{t_n} \\ &= M_{t_{n+1}} + \Theta\left(\Lambda_{t_n}, V_{t_n}\right) D_{t_{n+1}} - C\left(V_{t_n}\right). \end{aligned}$$

This leads to a semi-explicit form of the dynamics:

$$\left[\mathrm{Id} - \Theta\left(\Lambda_{t_n}, V_{t_n}\right)\Lambda_{t_{n+1}}\right] V_{t_{n+1}} = M_{t_{n+1}} + \Theta\left(\Lambda_{t_n}, V_{t_n}\right) D_{t_{n+1}} - C\left(V_{t_n}\right), \quad (3.8)$$

where Id denotes an $I \times I$ identity matrix.

Define the set of strictly positive investment proportions

$$\Xi = \left\{ \Lambda \in (\,0, 1\,]^{K \times I} : \sum_{k=1}^{K} \lambda_k^i = 1 - \lambda_0^i, \forall i \right\}.$$

We now address the question of whether the dynamics (3.8) is well-defined.

Theorem 3.1.1 (Existence and Uniqueness of Solutions) *Let $\Lambda_{t_n} \in \Xi$ for all $n \geq 0$.*
(i) For every $V_{t_n} \in [\,0, \infty\,)^I$ with $\sum_{i=1}^{I} V_{t_n}^i > 0$, there exists a unique $V_{t_{n+1}}$ that solves (3.8).
(ii) For every initial value $V_0 \in [\,0, \infty\,)^I$ with $\sum_{i=1}^{I} V_0^i > 0$, and a realization of the dividend-generating process $\delta(t)$, the discrete-time dynamics (3.8) generates a sample path $V_{t_n}, n = 1, 2, \ldots$.

Theorem 3.1.1 is essentially a restatement of Theorem 1 in Palczewski & Schenk-Hoppé (2010a). The difference is that the introduction of the risk free asset allows the additional constraint on aggregate wealth $\sum_{i=1}^{I} V_{t_n}^i = \frac{1}{c}$ for all n, where c denotes the same fixed consumption proportion for all investors, to be removed. The reasons for this are twofold. Firstly, as was demonstrated above, condition (3.7) that aggregate wealth has to satisfy is still in agreement with Walras' law but allows greater consumption flexibility. Secondly, the consumption process that will be used in our proposed model goes beyond a simple constant proportion for all agents. The precise formulation will be discussed in the following section. These two innovations to the basic evolutionary finance model allow the removal of the aforementioned additional aggregate wealth condition. The rest of the analysis, except for a few minor details regarding the investment strategies, is directly applicable to our case. The proof of Theorem 3.1.1 is given in the Appendix.

Theorem 3.1.1 ensures that the dynamics (3.6) can be written in explicit form:

$$V_{t_{n+1}} = \left[\mathrm{Id} - \Theta\left(\Lambda_{t_n}, V_{t_n}\right)\Lambda_{t_{n+1}}\right]^{-1}\left(M_{t_{n+1}} + \Theta\left(\Lambda_{t_n}, V_{t_n}\right)D_{t_{n+1}} - C\left(V_{t_n}\right)\right). \tag{3.9}$$

This representation will be used in the numerical simulation of the investors' evolution of wealth.

In this section we presented the basis of the stylized model of a financial market that we will be using in our study. Although the standard evolutionary model serves as a backbone of our formulation, we have also introduced a number of modifications that make our model different. These differences engender some important implications. We now move on to a discussion of one of the most important implications that result from our modifications to the general evolutionary finance model.

3.1.2 Bankruptcy

A common theme in evolutionary finance models is the analysis of long-term viability of investment strategies. Since this is the main avenue of examination, consumption decisions are usually relegated to a simple constant proportion of wealth that each investor consumes in each period. This does not allow the wealth dynamics to be contaminated or dominated by consumption decisions, as argued in Palczewski & Schenk-Hoppé (2010b), and allows for a precise analysis of the impact of investment strategies on the wealth dynamics.

Furthermore, popular agent-based papers, such as Chiarella, Dieci & He (2009), Hommes & Wagener (2009), and Chiarella, Dieci & He (2007), usually let the heterogeneity of investors to manifest itself in the form of different expectations about return distributions and the way agents estimate different market-related quantities. Investment decisions are either the outcome of expected utility maximization or simple heuristic behaviour like investing according to fundamentals or trend following. When it comes to the implementation of investment policies, agents are usually allowed to follow their investment program perfectly. There are no situations, in which investors have to forgo part of their intended investment in order to finance necessary consumption, or even sell part of their asset holdings in order to accommodate a temporary budget deficit. In essence, investors are allowed to borrow as much as they wish.

This kind of flexibility is rarely observed in reality. Virtually all kinds of investors face some practical constraints with regard to liquidity, minimum consumption, and sudden losses of large portions of their wealth endowments. This is particularly the case with large institutional investors, such as pension funds and life assurance companies, which dominate financial markets. This category of institutions have statutory obligations to pay out regular fixed annuity claims to retirees and policy holders. Even if their wealth endowments are not entirely depleted, an abrupt adverse movement in asset prices or periodic income streams can seriously hurt these large players, even to a point where they are unable to meet their liabilities and have to file for bankruptcy. Hence, it makes sense to establish some minimal bounds on consumption in order to limit the flexibility of economic agents and to bring these classes of models closer to real-world financial practice.

The modification of consumption to include a minimum required level and the introduction of bankruptcy are two of the main contributions of the current research project. These relatively simple amendments bring about an important problem that needs to be addressed. The existence of an arbitrary exogenous minimum consumption level m implies that agents may consume with greater intensity when their wealth endowments are not large enough and proportional consumption turns out to be below the minimum required level. This marks an important departure from existing literature focusing solely on proportional consumption. Unlike previous work in evolutionary finance, in the context of our model it is possible for the agents' wealth endowments to become negative: i.e. $V_{t_n} < 0$. This fact has important implications for market clearing due to the endogenous asset price specification.

It was shown above that the wealth endowment of each investor at any point in time comes from two sources: their stock market investments and their savings in the form of risk-free assets, i.e. $V_{t_n} = \Theta\left(\Lambda_{t_n}, V_{t_n}\right)\Lambda_{t_n} V_{t_n} + M_{t_n}$. Bankruptcy is defined as a situation, in which an investor's total wealth endowment is zero or less: $V_{t_n}^i \leq 0$. Since the general model of financial market dynamics that we formulated in the previous section is in discrete time, it is possible for a situation to occur, whereby the wealth $V_{t_n}^i$ of investor i at time t_n is positive, but the iteration of the dynamics from period t_n to t_{n+1} brings bankruptcy to agent i. Agent i's wealth $V_{t_{n+1}}^i$ at time t_{n+1} will almost surely turn out to be strictly negative, rather than precisely equal to zero, should a bankruptcy occur in such a setting.

Recall that the portfolio of investor i at time t_{n+1} in terms of the number of shares of risky asset k was given by (3.2) as:

$$\Theta_{k,t_{n+1}}^i = \frac{\lambda_{k,t_{n+1}}^i V_{t_{n+1}}^i}{S_{k,t_{n+1}}}, \quad k = 1, \ldots, K.$$

Similarly, since the risk-free asset has a constant price normalized to one, the amount of agent i's investment in the money market is given by:

$$M^i_{t_{n+1}} = \lambda^i_{0,t_{n+1}} V^i_{t_{n+1}}.$$

Both of these relations depend directly on $V^i_{t_{n+1}}$. The fact that $V^i_{t_{n+1}}$ is negative, implies that as of time t_{n+1} when it is discovered that investor i is bankrupt, this agent actually has short positions in the stock or money markets, or both. This represents a serious problem for market clearing in our closed-form economy.

Since investor i is now bankrupt, she is unable to cover her short positions. There would have been no way for other market participants to anticipate this, as at the time of the transaction t_n she was still solvent. Other agents find out that the market conditions at time t_{n+1} turn out to be unfavourable for the bankrupt agent and that some of the shares they hold are now worthless, since in essence the bankrupt investor had inadvertently sold more shares than she owns. Thus, investor i's trading partners will have to write off some assets as the aggregate wealth level in the economy declines.

Recall that asset prices in our model correspond to the aggregate amounts invested in the respective securities. The implications of such a bankruptcy scenario is that depending on the size of the outstanding short positions, the asset price $S_{k,t_{n+1}}$ might turn out to be zero or negative for some of the assets. Clearly, since negative prices are an economic impossibility, this will invalidate our market clearing mechanism, which relies on endogenous price formation.

This section outlines how such a situation will be resolved when it occurs and explains the relative advantages and shortcomings of the possible alternatives for handling this issue.

3.1.2.1 Removal of Investors Threatened by Bankruptcy

In order to avoid such a problematic situation, one possibility to handling bankruptcy that we propose is as follows. In case a situation similar to the one described above occurs, then the dynamics is temporarily stopped at time t_{n+1}, the bankrupt agent is removed from the market, and the dynamics is taken back to time t_n, where the remaining $I - 1$ agents repeat the calculation of their portfolios based on their trading strategies, asset prices, and aggregate level of wealth in the economy without the bankrupt investor i. Thus, a new set of portfolios $\Theta^i_{k,t_n}*$ are obtained, which still depend on the old strategies λ^i_{k,t_n} and wealth levels $V^i_{t_n}$, but also depend on a new set of asset prices $S_{k,t_n}*$. The asset prices change due to the exogenous way in which they are specified. After the removal of agent i, the sum inside the scalar product $\langle \lambda_{k,t_n}, V_{t_n} \rangle$ will range from $i = 1$ to $i = I - 1$, which will lead to lower asset prices and will change the agents' portfolios.

The intuition behind such a choice for handling bankruptcy is the intervention of a regulatory agency. An argument could be made that such an entity would observe the market and would remove all investors threatened by bankruptcy. The criterion for deciding which investor is at risk can be for example some capital adequacy ratio. Admittedly, the obvious weakness of such an argument is that it implies that the regulatory body has perfect foresight and can correctly forecast not just the wealth dynamics itself, but also all related quantities, such as the stochastic dividend generating process for instance. This is a rather strong assumption to make. Therefore, we now attempt to address the question of whether it is possible to avoid short positions altogether by finding the exact time when the wealth of the bankrupt investor will be exactly equal to zero.

3.1.2.2 Locating the Exact Time of Bankruptcy

The main purpose of specifying rules about how bankruptcy should be handled, such as the ones described in the previous section, was to avoid a situation, in which the bankrupt investor has short positions in the stock or money markets and is unable to cover them, thus preventing market clearing at time t_{n+1}. In order to avoid this disequilibrium, the bankruptcy procedure outlined above essentially amounts to checking if any of the investors has a strictly negative wealth at time t_{n+1} and if so, the step from t_n to t_{n+1} is re-done without the investor that would otherwise go bankrupt at time t_{n+1}.

While this solves the main problem, investor i is removed from the market while she still has a positive wealth endowment. Ideally, an investor would be declared bankrupt and removed from the market when her wealth is exactly equal to zero. Taking discrete steps makes it highly improbable for such a situation to occur naturally and that is why we had to resort to artificially removing the potentially bankrupt investor before her wealth becomes zero. Such an approach can lead to, at best, a rough approximation to the desired result. Furthermore, looking forward in time to check if an investor will go bankrupt at time t_{n+1} and going back one step to remove her from the market seems a bit unnatural and not very realistic. As discussed above, this would imply perfect foresight on the part of the regulator.

To address these weaknesses, we propose an improvement over the simple procedure described in the previous section. The basic idea is to check if any of the investors has a negative wealth at time t_{n+1} and if so, search for the exact moment in time between t_n and t_{n+1} where bankruptcy occurs – i.e. locate the point where wealth becomes equal to zero. Doing so will dispense with the need to artificially remove the bankrupt investor from the market, since when investor

i's wealth equals zero, the size of her stock and money market investments is also zero. In other words, having exhausted all her wealth, agent i will naturally cease her trading activities. The intuitive idea is that, theoretically, one should be able to search the entire interval between times t_n and t_{n+1} using as fine a discretization within the interval as necessary, so as to locate exactly the time at which the bankrupt investor's wealth endowment becomes equal to zero.

The implementation of the searching routine is a modification of the standard binomial search algorithm (see e.g. Kruse & Ryba (2000), ch. 7). If it is established that the wealth of at least one investor is negative at time t_{n+1}, the binomial search algorithm calls for checking the midpoint $t_n + \frac{(t_{n+1}-t_n)}{2}$ and observing the agents' wealth values. If at least one investor has a negative wealth endowment, then the exact point when her wealth becomes zero has occurred prior to the midpoint. Searching continues by moving to the left of the midpoint and checking all wealth values midway between times t_n and $t_n + \frac{(t_{n+1}-t_n)}{2}$ – i.e. at time $t_n + \frac{(t_{n+1}-t_n)}{4}$. Conversely, if all agents' wealth endowments are positive at time $t_n + \frac{(t_{n+1}-t_n)}{2}$, a bankruptcy has not yet occurred and searching continues by moving to the right of the midpoint and checking wealth levels midway between $t_n + \frac{(t_{n+1}-t_n)}{2}$ and t_{n+1} – i.e. at point $t_n + \frac{3(t_{n+1}-t_n)}{4}$. The binary search algorithm continues in an analogous fashion, operating on intervals of increasingly shorter lengths, until eventually the point at which the wealth of the bankrupt investor exactly equals zero is determined.

Unfortunately, there are a number of complications that arise if this method of handling bankruptcy is chosen. While there is sufficient evidence that such an approach could be mathematically valid, this has not been formally proven to date. Palczewski & Schenk-Hoppé (2010a) prove (Theorem 3, p. 920) that discrete-time dynamics similar, but not identical to (3.9) with an arbitrary length of the discrete time periods converge to a continuous-time limit. The introduction of the minimum consumption constraint, however, changes the structure of the underlying discrete-time model and convergence to a continuous-time limit is by no means guaranteed. We will not try to prove the continuous-time convergence of the discrete-time dynamics in (3.9) in this research project for two reasons. One is that even if we do so, there are other technical difficulties in implementing this approach. These are discussed in the following paragraphs. Additionally, even if we were able to resolve all these difficulties and prove continuous-time convergence, we believe there is a better method of handling bankruptcy that is more intuitive and requires considerably less computational effort. This approach is discussed in section 3.1.2.3.

In this paragraph we show that the standard searching procedure suggested above is not directly applicable to the problem we are facing, even if convergence to a continuous-time limit is ensured. The reason for this is that it can provide potentially misleading results due the endogenous nature of asset price formation. To see this, recall the wealth dynamics equation:

$$
\begin{aligned}
V^i_{t_{n+1}} &= \sum_{k=1}^{K} \frac{\lambda^i_{k,t_n} V^i_{t_n}}{\langle \lambda_{k,t_n}, V_{t_n} \rangle} \left(\langle \lambda_{k,t_{n+1}}, V_{t_{n+1}} \rangle + D_{k,t_{n+1}} \right) + \lambda^i_{0,t_n} V^i_{t_n} (1+r) \\
&- C^i_{t_{n+1}}.
\end{aligned}
$$

Suppose there were some investor with a negative wealth endowment at time t_{n+1}. We start by going back to t_n and taking only half a step to $t_n + \frac{(t_{n+1}-t_n)}{2}$. For each investor $i,\ i = 1 \ldots I$, a new wealth level $V^i_{t_n + \frac{(t_{n+1}-t_n)}{2}}$ will be calculated. In case there are no bankruptcies at this point, searching continues forward in time and tests wealth levels at the point $t_n + \frac{3(t_{n+1}-t_n)}{4}$. Notice that in the wealth dynamics equation above, the term describing the investors' portfolios at time $t_n + \frac{(t_{n+1}-t_n)}{2}$ – $\sum_{k=1}^{K} \frac{\lambda^i_{k,t_n+\frac{(t_{n+1}-t_n)}{2}} V^i_{t_n+\frac{(t_{n+1}-t_n)}{2}}}{\langle \lambda_{k,t_n+\frac{(t_{n+1}-t_n)}{2}}, V_{t_n+\frac{(t_{n+1}-t_n)}{2}} \rangle}$ – depends on the newly-calculated wealth $V^i_{t_n + \frac{(t_{n+1}-t_n)}{2}}$. After this midway rebalancing of the investors' portfolios, depending on the evolution of the dividend generating process and on the prevailing market conditions during the interval $[\, t_n + \frac{(t_{n+1}-t_n)}{2}, t_n + \frac{3(t_{n+1}-t_n)}{4}\,)$, there may or may not be any bankruptcies at time $t_n + \frac{3(t_{n+1}-t_n)}{4}$, irrespective of whether this was initially the exact point when the bankrupt investor's wealth reached zero. In other words, the rebalancing of investors' portfolios at each intermediate point during the search process changes the result of the search, because essentially, each additional step in the searching process means a new iteration of the dynamics during which anything could happen to the size of the wealth endowments depending on the stochastic asset payoffs. This fact invalidates the basic premise of the binomial search algorithm – it can no longer be claimed that if all wealth endowments are positive at time $t_n + \frac{(t_{n+1}-t_n)}{2}$ then the exact point of bankruptcy must be to the right of this time point. Hence, no meaningful, sorting procedure can be implemented.

Since the introduction of any kind of dependence on the number of steps taken during the search process will surely result in a biased search procedure, the only way to avoid this undesirable effect is to alter the standard binomial search algorithm. The modification that we propose is as follows. Similar to the approach taken in the previous section, we start by checking if the wealth endowment of any of the investors at time t_{n+1} is negative. If this is the case, we try to find the exact point in the interval from t_n to t_{n+1} when the bankrupt investor's wealth becomes equal to zero. This is done by going back to time t_n and taking half a step to $t_n + \frac{(t_{n+1}-t_n)}{2}$. Just as in the binary search, if no bankruptcy has occurred, only the interval to the right of the midpoint is considered. Conversely, if the wealth of any of the investors is already negative, then only the left half of the interval is searched. Unlike the standard approach, however, every time an additional step needs to be taken in any direction, the dynamics goes back to t_n and only then is a step of the necessary size taken. For example, if there happens to be no bankruptcy at time $t_n + \frac{(t_{n+1}-t_n)}{2}$, then we go back to t_n and using the asset prices, investment strategies and portfolios at that point, we take a step forward to time $t_n + \frac{3(t_{n+1}-t_n)}{4}$. Searching continues in this way until the moment when the wealth of any of the investors is found to be equal to zero. After that, portfolios are recalculated without the participation of the bankrupt investor, whose wealth is now zero, and the rest of the step to time t_{n+1} is taken.

The approach outlined above solves the inconsistency problem caused by the standard binary search algorithm in the context of our model of market dynamics. There are, however, a couple of technical issues that need to be addressed. Firstly, note that the bankruptcy procedure outlined in this section is possible, since we defined our dividend generating process in continuous time. The dividend paid is therefore scalable with respect to time. Recall that the amount of total dividends paid by asset k at the end of period $[\, t_n, t_n + (t_{n+1} - t_n)\Delta\,)$ was defined to be the integral of the dividend intensity over the appropriate time interval:

$$D_{k,t_n+(t_{n+1}-t_n)\Delta} = \int_{t_n}^{t_n+(t_{n+1}-t_n)\Delta} \delta_k\,(s)\,ds.$$

Using discrete time steps, the amount of dividend paid at time $t_n + (t_{n+1} - t_n)\Delta$ is $D_{k,t_n+(t_{n+1}-t_n)\Delta} = \delta_{k,t_n+(t_{n+1}-t_n)\Delta}\,(t_n + (t_{n+1} - t_n)\Delta - t_n)$. This allows the appropriate amount of dividends to be included in the equations governing the wealth dynamics regardless of the size of the steps during the search routine. In order to find the exact point of bankruptcy, we need the continuous time properties of the dividend generating process. Locating the exact point of bankruptcy, on the other hand, implies that there will be no outstanding short positions that cannot be covered or removal of investors with positive wealth endowments. Therefore, market clearing will be ensured.

The second issue regarding the feasibility of the search algorithm proposed above concerns the definition of the agents' investment strategies. Unlike dividends, the investment strategies λ^i_{k,t_n} are defined in discrete time, since agents only trade at discrete points in time. This is potentially a problem. In order to take the step from t_n to $t_n + (t_{n+1} - t_n)\Delta$, both λ^i_{k,t_n} and $\lambda^i_{k,t_n+(t_{n+1}-t_n)\Delta}$ must be known at time t_n. This fact is of no cause for concern when the length of Δ coincides with the trading times at which investors rebalance their portfolios, but is problematic when $\Delta < 1$. Since only λ^i_{k,t_n} and $\lambda^i_{k,t_{n+1}}$ are defined in the model, we have to do the same for the times when intermediate values $\lambda^i_{k,t_n+(t_{n+1}-t_n)\Delta}$, $\Delta < 1$ are required. The easiest way to do so is by means of a simple linear interpolation:

$$\lambda^i_{k,t_n+(t_{n+1}-t_n)\Delta} = \frac{t_{n+1} - t_n + (t_{n+1} - t_n)\Delta}{t_{n+1} - t_n}\lambda^i_{k,t_{n+1}} + \frac{t_n + (t_{n+1} - t_n)\Delta - t_n}{t_{n+1} - t_n}\lambda^i_{k,t_n}. \tag{3.10}$$

This definition ensures that the budget shares always add up to the same number. It also guarantees that the intermediate values for the investment strategies $\lambda^i_{k,t_n+(t_{n+1}-t_n)\Delta}$ meet all necessary conditions satisfied by λ^i_{k,t_n} and $\lambda^i_{k,t_{n+1}}$, which are required so as to ensure matrix invertibility when trying to solve the equation governing the wealth dynamics.

One obvious disadvantage of the method for handling bankruptcies described in this section is its considerably larger computational cost. Compared to the simple removal of the investors threatened by bankruptcy, which only requires going back to time t_n once, the need to perform a search over the intermediate times in the interval $[\, t_n, t_{n+1}\,)$ significantly increases the necessary processing time. It is natural to ask if such an increase in cost is warranted by increases in the performance of the model in terms of producing significantly different dynamics. While the approach in this section leads to definite improvements in accuracy when the length of the interval is relatively large, the extent of its benefits are questionable for closer-spaced periods. In fact, for a sufficiently small size of the time step, the two methods described above will lead to similar results, thus removing the need for the additional computations required by the searching algorithm.

3.1.2.3 Allocating the Aggregate Loss Resulting from Bankruptcy

The two methods for handling cases of bankruptcy described above have one thing in common. They both avoid situations, in which the bankrupt investor has any outstanding short positions in the stock or money markets at the time insolvency is declared. This certainly improves tractability as the solvent investors do not have to personally bear the consequences of an investor going bankrupt. Due to approximation errors, however, the first method described above results in a decrease in aggregate wealth in the market, owing to the premature removal of the would-be bankrupt investor, who still has a positive wealth endowment at the time she ceases her trading activities.

Nevertheless, from a practitioner's viewpoint, the two methods for handling bankruptcy described above may seem highly idealized. In fact, it is not at all uncommon for large investment entities to face insolvency in real-world financial practice. The financial crisis of 2007-2008 saw several large institutional investors and even investment banks file for bankruptcy. The consequences of such failures are significant – in many cases regulators are too fearful of systemic risk and must resort to costly bailouts. Regardless of whether the troubled financial institution is saved or left to fail, its losses are quickly propagated through the global financial system.

That is why in this section we present a third, more realistic alternative for handling bankruptcy cases. Instead of trying to determine a point in time where investor i's wealth becomes zero, we consider only the ordinary discrete time steps at which agents rebalance their portfolios. If the wealth endowment of any of the investors is negative at time t_{n+1}, then she is simply allowed to go bankrupt. This is followed by a "bankruptcy procedure" in the legal sense of the word, in which the insolvent agent's wealth is set equal to zero and she is removed from the market. Any outstanding short positions will bring a loss to the other market participants who are in possession of shares of the risky assets affected by the bankruptcy.

To see how this would all work out, consider the case when there are only two investors in the market, trading with each other – i.e. $I = 2$. Assume that the sum of their wealth endowments at time t_{n+1} is strictly positive. In other words: $V^1_{t_{n+1}} + V^2_{t_{n+1}} > 0$. This condition ensures there is at least one solvent agent remaining, so that the dynamics can continue after settling any bankruptcies. Suppose it has transpired at time t_{n+1} that the second investor had gone bankrupt and her wealth is now negative: $V^1_{t_{n+1}} > 0$, $V^2_{t_{n+1}} < 0$.

At this point the dynamics is temporarily stopped and a bankruptcy procedure is initiated. Firstly, the current values of the investors' positions in both the stock and the money markets are recorded before starting the procedure. This will lead to positive values for the net positions of the solvent investor in the stock and money markets respectively – $M^1_{pre} > 0$ and $\Theta^1_{k,pre} > 0$. The subscripts t_{n+1} have been suppressed to avoid overly cluttered notation. This will be the case for the rest of this section. Similar records are made for the insolvent investor. Her net position in the market for risky assets will be short, however, because of the negative wealth: $M^2_{pre} = \lambda^i_{0,pre} V^i_{pre} < 0$ and $\Theta^2_{k,pre} = \frac{\lambda^i_{k,pre} V^i_{pre}}{S_{k,pre}} < 0$.

Next, the bankruptcy procedure is initiated: the wealth (and hence the portfolio) of the insolvent investor is set to zero and she is removed from the market. As the only remaining investor, agent 1 now holds all available assets in the market. That is: $\Theta^1_{n+1} = (1, \ldots, 1)$ and $\Theta^2_{n+1} = (0, \ldots, 0)$. A similar approach is used for dealing with the situation in the money market. The money market position of investor 2 is set to zero. The position of the remaining investor is defined to be the sum of the money market positions of the two investors before the bankruptcy procedure was initiated. Since the net position of investor 2 was negative and cannot be recouped, agent 1 suffers a loss equal to the magnitude of the bankrupt investor's short position. Expressed symbolically, we have: $M^1_{n+1} = M^1_{pre} + M^2_{pre} < M^1_{pre}$ and $M^2_{n+1} = 0$. Notice that in this case, the aggregate wealth in the market is decreased, but unlike the case with the premature removal of the investors threatened by bankruptcy, this happens through a direct loss to the solvent investors.

The above simple illustration with two agents can be easily extended to cases with multiple investors. The difference is that now all the remaining agents will hold all the available assets in the market – $\sum_{i=1}^{I-1} \Theta^i_{k,n+1} = 1$. Additionally, the losses caused by the bankrupt investor's short positions will be spread among all solvent agents. Here an interesting question emerges: should the losses be shared in proportion to the size of the wealth endowments or based on the magnitude of investors' positions in the affected assets. The intuition behind the first possibility is that the size of an agent's stock and money market positions is directly proportional to their wealth. Therefore, an investor with a greater exposure to the money market should bear a larger share of the losses, since she would have engaged in transactions with the bankrupt agent with a greater probability. This is so because the more shares an investor owns, the greater the chance that at least some of these would be exposed to counterparty risk with respect to the bankrupt agent. There is a limited supply of the risky asset, so to amass a large holding of the asset, more shares would have been obtained from the currently bankrupt investor, relative to an investor that holds just a few shares of the risky asset.

The latter, however, is a flawed approach since disregarding the size of the investors' positions in the respective assets brings the risk of artificially inducing short positions for some of the solvent investors if they have to bear a disproportionately large share of the losses. Therefore, we opt to spread the losses resulting from bankruptcy among the remaining solvent investors on the basis of the magnitudes of their outstanding positions in the affected risky assets. This ensures market clearing while safeguarding against artificially induced short positions among the solvent agents. The merit of this approach is best illustrated through an example.

Example 3.1.2 *Let the model be populated by four investors and contain three risky assets: $I = 4$, $K = 3$. Assume at time t_n investor 4 has a positive money market account balance, a positive wealth endowment and no short positions. The dynamics (3.5), however, suggests that at time t_{n+1} her wealth will be negative. Further, assume the agents' positions as of time t_n are as follows:*

Table 3.1: Bankruptcy Procedure - Illustration

$Investor$	M^i_{pre}	$\Theta^i_{1,pre}$	$\Theta^i_{2,pre}$	$\Theta^i_{3,pre}$
1	M^1_{pre}	1.5	0.5	0
2	M^2_{pre}	0.5	0.5	0
3	M^3_{pre}	0	0	1
4	2	-1	0	0

Here, the money market positions M^i_{pre} are before dividends and consumption and that is why positive values for the bankrupt investor can be achieved. Let the price of the asset affected by investor 4's bankruptcy at time t_{n+1} be $S_{1,t_{n+1}} = 3$. Then, clearly investor 4 is bankrupt at time t_{n+1} as his total wealth endowment will be equal to $V^4_{t_{n+1}} = 2 - 3 < 0$. If the losses caused by investor 4 are distributed according to the size of the wealth endowments $V^i_{t_{n+1}}$, then investor 3 would surely end up with an artificially produced short position in asset 1. Depending on relative wealth, the same might be true for investor 2 as well. Instead we choose to distribute both the stock market and money market positions of the bankrupt investor to the remaining solvent investors in proportion to the size of their portfolios.

After the removal of investor 4 from the market, the net supply of asset 2 will be brought back to one by taking away $\left(\frac{1.5}{2}\right) 1$ from the first investor's shareholdings and $\left(\frac{0.5}{2}\right) 1$ from the second agent. Following this, investor 1 will hold 75% of asset 1 and investor 2 will be in possession of 25% for a net supply of one. It is important to note that the price of asset 2 remains unaffected by the bankruptcy procedure and thus fully capture the negative effects of the short sales.

Even though the procedure outlined above is the most logical and robust approach to dealing with bankruptcy, it is not without shortcomings. A problem will occur of the size of the short positions exceeds the total value of the long positions. Under these circumstances, there is nothing to prevent the asset price from being negative. This is demonstrated in the following counterexample.

Counterexample 3.1.3 *Let $I = 4$, $K = 1$ and assume investment strategies and wealth endowments as given in the table below:*

Table 3.2: Bankruptcy Procedure - Counterexample

$Investor$	λ^i_{1,t_n}	$V^i_{t_n}$	$\lambda^i_{1,t_n} V^i_{t_n}$
1	0.9967	-0.01	-0.01
2	0.9967	-0.01	-0.01
3	0.5	0.0064	0.0032
4	0.5	0.0215	0.0108
S_{1,t_n}			-0.006

In this case the short positions of the first two investors dominate the whole market and push the price to negative values. Since this is a very rare occurrence, we discard it as a market failure. In our numerical simulation in order to safeguard against negative asset prices, if such a situation is observed, we set the bankrupt investors' wealth endowments to zero and recalculate the asset price using the investment strategies and wealths of the remaining solvent investors only.

In addition to being more realistic, the bankruptcy procedure laid out in this section also has the added benefit of being computationally the least demanding of the three.

3.1.3 Summary

In this section we presented a general model of a stylized financial market. Risky assets as well as a risk-free asset are traded at discrete points in time. We thus formulate a discrete-time dynamics for the evolution of investor wealth. In the limit as the length of the time periods approaches zero, the dynamics coincides with a popular approach of modeling self-financing portfolios frequently used in mathematical finance.

Despite the wealth dynamics being formulated in discrete time, dividend payments were defined in terms of continuous intensities. This approach ensures scalability and consistence of dynamics with time intervals of different length. It also allows the exact time of bankruptcy to be located.

Each investor type was described by an arbitrary investment strategy based on heterogeneous beliefs. Asset prices are obtained endogenously and implicitly reflect the price impact of all market participants. The consumption good serves as a numeraire: all dividend payments payment are paid in terms of the consumption good. One of the innovations of our model is to enable the intertemporal transfer of consumption by allowing agents to postpone consumption by investing in a risk-free asset whose price is also normalized to one. The other main innovation over existing evolutionary and agent-based models is the inclusion of a more general consumption process, which allows different intensities of consumption. This more realistic approach introduces the risk of bankruptcy. The notion of bankruptcy has some important connotations for market clearing. Such a situation would entail short positions in the stock or money markets, which the insolvent agent is not able to cover, leading to the presence of more shares of an asset in the economy than the level of aggregate wealth can support.

The problem of bankruptcy is introduced in our model because of the modification of consumption, which guarantees a minimum level of outflows in each time period regardless of the size of an agent's wealth endowment. This can accelerate the rate at which an investor loses her wealth, particularly during times of decreasing dividend income.

We reviewed three possible solutions for addressing the situation, in which at least one investor ends up with a non-positive wealth endowment. The first one checks if an investor will be insolvent in the next time step and if so removes him from the market ahead of the actual bankruptcy. The second alternative refines this idea by searching for the exact point between the two endpoints of the interval in time where the bankrupt investor's wealth becomes equal to zero. Once this point is found, the removal of the insolvent agent from the market happens naturally and all markets clear without any additional intervention. Finally, we discussed a more realistic situation, in which the bankrupt agent is left to fail and the remaining solvent agents share the losses caused by her short positions in proportion to the relative sizes of their asset holdings.

In the next sections we continue to extend the base case model by specifying in more detail some of its components that were assumed arbitrary in the previous section. We begin by defining the modified consumption process in section 1.2. We present some relevant intuition as to why this adjustment to simple proportional consumption is necessary in our case. We also mentioned in the previous section that in contrast to the traditional mathematical finance framework of specifying an exogenous stochastic asset price process, randomness in our model stems from two sources: the uncertain asset payoffs and the interaction between investors.

Both of these were mentioned as general processes above. Consequently, in the following sections we address them in more detail. Randomness in the asset payoffs comes from uncertainty in the economy, in which companies operate. This uncertainty makes their earnings risky. Since dividends are a function of corporate earnings, the same applies to them as well. Section 1.3 describes two similar ways of modeling this uncertainty in the economy, which contain all necessary attributes of the business cycle.

The second source of randomness is the market interaction between investors. The reason why this occurs is the specification of a heterogeneous group of economic agents, in which each investor category trades according to its specific beliefs about the market. Each investor type is described fully by the investment strategy it follows. In the agent-based literature these strategies can be the result of either standard utility maximization or simple heuristic strategies consistent with bounded rationality. Section 1.5 explains our proposed investment strategies that investors will follow in our stylized market. These include the general classes of fundamentalists and chartists, as well as additional, more complex investment types.

The latter reflect the fact that investors have multiple opportunities to learn about the business cycle over time through their investment experience and will adjust their behaviour accordingly. Therefore, we also need to describe the way investors learn within the context of the model. Some of these more complex strategies are fundamental in nature and rely on dividend yields. Others reflect the behaviour of a value investor who invests on the basis of perceived deviations from fundamental values. Hence, before discussing the investment strategies in detail, section 1.4 describes the Monte Carlo algorithm we use for the computation of fundamental values.

3.2 Guaranteed Payment of Minimum Liabilities

As discussed briefly in the previous section, the lack of a minimum consumption threshold in existing agent-based models allows investors to freely follow their desired trading strategy without regard to their success in the market or the size of their wealth endowment. Consumption decisions are kept separate from investment decisions. A similar situation exists in popular evolutionary finance models, where the same proportion of wealth is consumed by all agents. The argument that existing papers in these fields make is that in order to produce results relating to the relative success of different trading strategies, investment decisions must not be contaminated by the effects of consumption.

Indeed it can be seen in Palczewski & Schenk-Hoppé (2010b) that in their stylized continuous-time model without a risk-free asset and with time-invariant strategies, a lower consumption proportion directly translates to superior performance as measured by the size of the agents' wealth endowments. Within the context of their model, assuming investor i has a lower consumption rate – $c^i < c^j$ – the following result follows:

$$\frac{dV^i(t)}{V^i(t-)} - \frac{dV^j(t)}{V^j(t-)} = \left(-c^i + c^j\right) dt > 0.$$

That is, the instantaneous growth rate of the investor with a lower consumption proportion strictly dominates the growth rate of the other investor's wealth.

While the assumption of a uniform consumption proportion definitely aids in the analysis of the relative performance of investment strategies, it is difficult to justify it taking into consideration the substantial asset-liability constraints that large institutional investors face in practice. A fund which is able to generate above-average risk adjusted returns may still be in a underfunded position depending on how large its liabilities are. Therefore, in the case of pension funds and other large institutional investors, it is the relative balance between their assets, which are a function of their investment success, and their liabilities that ultimately determines how viable they are. Hence, the additional constraints imposed by the specific framework in which this group of investors operates must also be directly included in the analysis of agent interactions. Moreover, this does not contradict the main performance criterion used in evolutionary finance: the size of the wealth endowment.

To this end, we specify a two-tier consumption process. The consumption rate above a certain floor is the same proportion for all investors. As soon as a predetermined minimum level is reached, however, consumption cannot fall below this threshold. That is, investor i's total consumption, which takes place at the end of of the time interval $[\, t_n, t_{n+1}\,)$, is given by:

$$C^i_{t_{n+1}} = \max\{m, cV^i_{t_n}\}\,(t_{n+1} - t_n)\,, \tag{3.11}$$

for some predetermined minimum level m and a constant c, with $0 < c < 1$ for all investors.

From the viewpoint of institutional investors, consumption defined in this way can be interpreted as the regular payments made to retirees and policy holders. These pay-outs constitute a fixed proportion c of the total capitalization of the fund when it possesses sufficient assets, but cannot fall below the minimum guaranteed payment m regardless of the amount of the fund's wealth. As was discussed above, consumption is financed out of the money market account and in the case of insufficient funds, the shortfall is covered by the sale of securities from the institution's long-term portfolio in the stock market. This situation is known as underfunding.

When the minimum level is activated, investors begin to consume at a greater intensity relative to the size of their wealth endowment. Unless a favourable sequence of asset payoffs occurs, an investor consuming at this higher rate will continue losing market share as he is forced to liquidate more of his risky asset investments. This, in turn makes it increasingly more difficult for her to recover, since she will be in possession of fewer assets capable of generating cash flows. This faster wealth erosion is the mechanism that allows the possibility of bankruptcy.

Both the interconnectedness of consumption and investment decisions as well as the endogenous asset price formation are features of our model that are common in the existing literature in the established field of dynamic stochastic general equilibrium models. This class of models differs greatly from our work, however, since investment-consumption decisions are the product of expected utility maximization under the assumption of rational beliefs. Such an approach has a long tradition in tackling optimal investment-consumption decisions and asset allocation (see e.g. Merton (1971)), as well as asset-liability management and numerous other applications (see Stokey et al. (1989)). A significant "blow" to the assumption of full rationality underlying stochastic general equilibrium models was illustrated in Blume & Easley (2006), who showed that the market selection hypothesis fails in incomplete markets.

3.3 Specification of Uncertainty in the Economy

Uncertainty in the economy stems from two sources: the business cycle and random short-term fluctuations. Recessionary periods are followed by recoveries, leading to intense economic growth and slowing down again. In the economic literature, especially in the field of general equilibrium models, this uncertainty is said to be caused by random changes in the state of technology. Since new technological breakthroughs are impossible to model precisely, usually the development of the state of technology is assumed to be governed by a stochastic process. The production opportunities that will be available in the future depend on the accessible technology and this causes the different phases of economic growth.

Since company earnings, and hence asset payoffs, are a function of the state of the economy, and more specifically by the state of the particular industry that a company operates in, in our model this uncertainty in the economy is incorporated in the exogenous process driving the amount of dividends paid by each risky asset. As discussed above, this process evolves in continuous time and is specified using a vector of intensities $\delta(t)$. The intensity represents information that investors can use in order to formulate their investment strategies. Actual trading and the payment of dividends, however, take place only at arbitrary discrete points in time. The amount of total dividends paid by asset k at the end of the period $[\, t_n, t_{n+1}\,)$ is obtained by integrating the dividend intensity over this time period:

$$D_{k,t_{n+1}} = \int_{t_n}^{t_{n+1}} \delta_k(s)\, ds. \tag{3.12}$$

The magnitude of the dividend generated by the risky asset is dependent on the evolution of the economy over time.

A vast amount of literature exists concerning the modeling of the business cycle. The most popular modeling approach is the use of regime-switching models. There exists some disagreement as to the number of distinct regimes of an economy within the setting of this class of models. However, in our work for the purposes of simplicity we opt for a simple two-state model of the economy. We cover two broad possible states – periods of prosperity and moderate economic growth, followed by periods of economic contraction and recession. These regimes can also be applied to the particular industries in which companies operate. Since the profitability of companies in different industries is not perfectly correlated with the state of the general economy, applying this two-regime modeling approach to each industry is a flexible way to allow the study of the impact of potential diversification benefits to the behaviour of the market dynamics in the context of minimum consumption guarantees.

Even in this simple two-regime case, a substantial amount of debate exists in the literature with regard to the correct modeling of the two phases. Kim & Nelson (1999) argue that the switches between recessions and prosperity tend to be asymmetric, with recessions happening much more suddenly and having a much more pronounced immediate effect. Considerable discussion is given as to whether the shocks that bring about periods of recessions and prosperity are temporary or permanent. Following the influential work by Hamilton (1989), who views the switching of regimes as an autoregression, whose parameters are produced by a discrete-time Markov process, Durland & McCurdy (1994) follow a similar approach and find that the switches between the different states are dependent on their duration.

Due to the prevalence of similar modeling approaches in the literature, we follow a Markovian approach as well. The switches between the two states of each industry are governed by a two-state continuous-time Markov process, with exponentially distributed waiting times between switches. Each state is characterized by its own mean dividend value $\mu(t)$. This variable can take on two values, corresponding to the mean dividend values in each regime: the average dividend in the prosperity regime of moderate economic growth is denoted by $\overline{\mu}(t)$, while the average for the recessionary regime is $\underline{\mu}(t)$, where $\overline{\mu}(t) > \underline{\mu}(t)$. The recessionary regime is characterized by a lower mean-reverting value, as companies cut back on discretionary cash outflows during economic contraction when there is a lack of profitable production opportunities.

Existing literature is not unanimous about the distribution of the durations of recessions and prosperity periods or their relative duration. The main debate is whether the distribution of durations follows a power or an exponential law. In their study, Ausloos et al. (2004) find evidence in favour of both. Thus, the question of the distribution of the durations of the different regimes of the business cycle remains open. The authors, however, find that periods of prosperity tend to last roughly six times longer than recessions.

Based on the above literature we choose to model the regimes of each industry in the following way. The evolution of the variable $\mu(t)$ follows a continuous-time Markov chain. The waiting times between the switches constitute an exponentially distributed random variable. Since we are working with discrete time periods, we need to implement a suitable approximation to the total number of regime switches in the time span under study. A convenient way to accomplish this is by comparing an exponentially distributed random variable to a value drawn from a uniform distribution to see if a regime switch occurred during each of the discrete time periods. This is a standard numerical approach for conducting Bernoulli-type experiments. This procedure provides a reasonable approximation to the number of regime switches, as with a sufficiently small time step, the probability of having more than one switch in a period becomes very small.

We note that for an exponentially distributed random variable X, the probability of the variable exceeding a certain value x is given by:

$$P(X > x) = 1 - F(x) = 1 - \left(1 - e^{-\lambda x}\right) = e^{-\lambda x},$$

where λ is the intensity parameter and $F(\cdot)$ denotes the cumulative density function of the exponential distribution. A switch would occur if in any one period the uniformly distributed random variable exceeds the exponentially distributed one. In other words, using the result above, the discrete-time approximation to the conditional probability of staying in the same regime (for instance, prosperity) is given as:

$$P\left(\mu_{t_{n+1}} = \overline{\mu} \mid \mu_{t_n} = \overline{\mu}\right) = e^{-\lambda(t_{n+1}-t_n)}. \tag{3.13}$$

The conditional probability of a regime switch is then simply $1 - e^{-\lambda(t_{n+1}-t_n)}$. The average time the process spends in each regime is the first moment of the exponential distribution – $\frac{1}{\lambda}$.

The literature discussed above suggests that recessions tend to exhibit a much more pronounced rate of change in an economy's growth. These periods, however, usually do not last as long as prosperity periods. To account for this finding, the waiting times for each state are drawn from two exponential distributions with different rate parameters. The regime switches constitute the unobservable part of the dividend intensity. Investors are given no information about the state of each industry. In other words, the regime switching process forms a hidden Markov model (see e.g. Elliott et al. (1995)).

Using the regimes as a general framework, the second component of the dividend intensity deals with shorter-term random fluctuations within each state caused by multiple, and often unobservable, factors of lesser significance. To account for these noise factors, we model the asset's state-dependent dividend generating process by means of mean reverting stochastic differential equations, in which the mean reverting values correspond to the average dividend values in the two states of each industry. This approach is in agreement with existing literature in the field. Two alternative formulations for the dividend generating process are considered below.

3.3.1 Mean-Reverting Square Root Process

The mean-reverting square root process is also popular in the financial economics literature as Cox-Ingersoll-Ross process (CIR). The first process that we use to model the risky asset payoffs is precisely this mean-reverting square root process, which we use to specify the evolution of dividend intensities over time. The dividend intensity process, which was given as arbitrary in the previous section, is now given as follows:

$$d\delta_k(t) = \alpha \left(\mu_k(t) - \delta_k(t)\right) dt + \beta\sqrt{\delta_k(t)}dW(t). \tag{3.14}$$

In the above, α and β are strictly positive real-valued parameters, and $W(t)$ is a standard Wiener process. Unlike the regime switching process, which remains hidden, investors know that dividends follow the process specified in equation (3.14). The equations specifying the intensity are the observable part of the dividend generating process.

The instantaneous change in dividend intensity is obtained as the sum of a drift term and a square-root diffusion term. The drift $\alpha\left(\mu_k(t) - \delta_k(t)\right)$ measures the speed of adjustment toward the equilibrium mean-reverting dividend value. The diffusion term $\beta\sqrt{\delta_k(t)}$ ensures that dividends cannot become negative as long as α and β are non-negative. It is a well-known fact (see Feller (1951), p. 174, property (iii)) that the CIR process is strictly positive if the Feller condition $2\alpha\mu_k(t) > \beta^2$ is satisfied.

We discretize equation (3.14) by using Milstein's scheme (see e.g. Kloeden & Platen (1999), section 10.3, p. 345). The discrete-time version of (3.14) then becomes:

$$\begin{aligned} \delta_{k,t_{n+1}} &= \delta_{k,t_n} + \alpha\left(\mu_{k,t_n} - \delta_{k,t_n}\right)(t_{n+1} - t_n) + \beta\sqrt{\delta_{k,t_n}}\left(W_{t_{n+1}} - W_{t_n}\right) \\ &+ \frac{1}{4}\beta^2\Big(\left(W_{t_{n+1}} - W_{t_n}\right)^2 - (t_{n+1} - t_n)\Big). \end{aligned} \tag{3.15}$$

Finally, the total dividend paid by asset k at the end of the period $[\, t_n, t_{n+1}\,)$

is approximated by:

$$D_{k,t_{n+1}} = \delta_{k,t_{n+1}} \left(t_{n+1} - t_n \right). \tag{3.16}$$

It is evident from the $(t_{n+1} - t_n)$ terms in (3.13) and (3.16) that dividends are scalable with respect to time. As noticed in the previous section, this is an important property that safeguards the model against discrepancies in the amount of dividends paid caused by changing lengths of the time periods.

3.3.2 Geometric Ornstein-Uhlenbeck Process

Another widely used modeling approach in describing the stochastic evolution of the unlevered equity value as well as the valuation of investment projects over time is the geometric Ornstein-Uhlenbeck process (OU) (see e.g. (Dixit & Pindyck 1994)).

Our second formulation of the dividend generating process uses a geometric Ornstein-Uhlenbeck process to model the random evolution of the dividend intensities:

$$d\delta_k(t) = \alpha \left(\mu_k(t) - \delta_k(t) \right) \delta_k(t) dt + \beta \delta_k(t) dW(t). \tag{3.17}$$

In the context of this mean-reverting formulation, the future dividend values are lognormally distributed with a linearly growing variance. Again, we use Milstein's scheme (Kloeden & Platen 1999) to obtain the corresponding discrete-time version. In the case of the geometric Ornstein-Uhlenbeck process, the discrete-time approximation to (3.17) is:

$$\begin{aligned} \delta_{k,t_{n+1}} &= \delta_{k,t_n} + \alpha \left(\mu_{k,t_n} - \delta_{k,t_n} \right) \delta_{k,t_n} \left(t_{n+1} - t_n \right) + \beta \delta_{k,t_n} \left(W_{t_{n+1}} - W_{t_n} \right) \\ &+ \frac{1}{2} \beta^2 \delta_{k,t_n} \left(\left(W_{t_{n+1}} - W_{t_n} \right)^2 - \left(t_{n+1} - t_n \right) \right). \end{aligned} \tag{3.18}$$

Both the approximation and the original process possess an interesting property that is readily seen from the above. For a starting dividend value of zero, this dividend generating process remains at zero in all subsequent periods – i.e. zero is a fixed point. This fact has some important implications for the calculation of the risky asset's fundamental value. These will be discussed in more detail when we present the algorithm for the computation of fundamental values.

3.4 Fundamental Values

This section discusses in more detail the calculation of fundamental values within the context of our model. Fundamental values here will be understood as the perceived intrinsic value of a risky asset, based on the expected discounted stream of future cash flows to the investor that it will generate. While interesting in its own right, the computation for fundamental values is strictly necessary in our modeling framework, since the investment strategies used by agents depend heavily on fundamental value estimates and perceived discrepancies between asset prices and fundamental values.

Because our dividend generating process was specified in terms of both an observable and an unobservable component, the algorithm for the calculation of fundamental values consists of two steps. Firstly, a number of fundamental values are computed for each risky asset. The precise quantity depends on the number of regimes of each industry. These calculations represent the fundamental value an investor would assign to the asset if he knew with certainty the current regime of each industry. As discussed above, the regime switches are governed by a Markov process but the states of each industry remain hidden. This is the first source of uncertainty for any investor attempting to calculate a fundamental value estimate.

The second source of uncertainty are the random fluctuations around the mean-reverting dividend values. To cope with this, a Monte Carlo algorithm is implemented. This involves the calculation of fundamental values for each element of a vector of starting dividends. Since our model deals with two general regimes – prosperity and recession – each risky asset will have two vectors of fundamental values corresponding to the asset's fundamental value if the regimes were know with certainty. Each element in the vectors corresponds to a different starting dividend value. In the case of two existing regimes in each industry, the true fundamental value for each asset will be between the two values calculated under full knowledge about the regimes. Thus, in this special case, the fundamental values under certainty can be interpreted as boundaries for the true fundamental value. The precise location of the true fundamental values between the boundaries depends on the estimated conditional probabilities of each regime. This brings us to the second stage of fundamental value calculation.

The second stage in the calculation of fundamental values is the filtering of the state process based on the observations of the dividend realizations. This is implemented by means of Bayesian updating. As a result, at each point in time investors calculate the conditional probability of being in the two states of each industry based on the observations of the dividend process. Once this is done, the final step is to calculate the true estimated fundamental values by applying the probabilities to the relevant fundamental value under certainty to come up with an estimated true fundamental value that lies somewhere in between the two limits calculated during the first stage, assuming there are only two regimes in each industry.

3.4.1 Fundamental Values under Certainty

In order to calculate the fundamental value points under full information about the regimes, we use the discounted-cashflows methodology. Since dividends are generated by a random process, a Monte Carlo valuation framework is implemented. Additionally, we also have two regimes, which determine the average dividend values. These regimes, however, remain unobservable to the agents, who can only see the value of the dividend but are given no information about the underlying economic state.

Therefore, each agent can use the available dividend information and assign a value for the risky asset equal to the sum of discounted cashflows, but these calculated values can only represent fundamental values under full information about the industries' regimes. As discussed above, in the case of two regimes, these calculated values will form an upper and a lower boundary for the fundamental value under uncertainty about the regimes. This is so because there is a vital piece of information missing – investors have perfect knowledge of the dividend generating process, but no information about the current state of each industry. That is to say, given perfect information about the economic cycle, each investor would agree that only one of those two computed boundaries is the true fundamental value because each agent knows which state each industry is currently in and what state can be expected in the near future. So, for instance in the case of a prosperous economy, the upper boundary will be the true fundamental value, whereas in the case of recession, this will be the lower boundary.

However, in reality investors rarely agree on what to expect from the economy in the future and it is not uncommon for different analysts to come up with significantly different fundamental value estimates. Consequently, each agent in our model will have to estimate the probability of being in each regime and then apply these to the two calculated fundamental values under certainty in order to compute a perceived fundamental value for each risky asset, which will be somewhere in between the two boundaries that represent the fundamental values on condition that agents know the underlying regime with certainty.

For each state we compute a vector of fundamental values under certainty by taking expectations of the sum of discounted cashflows. In the most general case, for an arbitrary starting point in time t and a starting dividend δ_0, the fundamental value is given by:

$$\begin{aligned} FB(t,\delta,\mu) &= E_t\left[\sum_{n,t_n\geq t}^{\infty} D_{t_n} e^{-r(t_n-t)}\right], \\ & \mu(0)=\mu\in\{\overline{\mu},\underline{\mu}\};\ \delta_0=\delta. \end{aligned} \tag{3.19}$$

Since, however, investors receive aggregate dividend payments and rebalance their portfolios based on their trading strategy at discrete points in time $\{t_n\}$, a situation where a fundamental value has to be estimated for some intermediate point between two successive dividend payment dates never occurs. The calculation of a fundamental value for a specific starting dividend δ_0 then reduces to the following:

$$\begin{aligned} FB\,(\delta,\mu) &= E\left[\sum_{n=1}^{\infty} D_{t_n} e^{-rt_n}\right], \\ &\mu\,(0) = \mu \in \{\overline{\mu}, \underline{\mu}\};\ \ \delta_0 = \delta. \end{aligned} \tag{3.20}$$

The dependence of fundamental values on δ and μ can be better explained via a discussion of the algorithm used in their calculation. At each point in the simulation, the dividend generating process can take on values in a continuous range. The fundamental value is concerned with expected future dividends. Therefore, we discretize the range of possible starting dividend values, generate many dividend realizations from that point on and calculate expectations of these values, appropriately discounted. Therefore, at each time point t_n the fundamental value depends on the starting dividend value, which was observed. The dependence on μ is implicit since in order to generate dividend intensities we need to know the parameter μ. The latter is never observable, so we generate two fundamental value boundaries under the assumption that $\mu = \overline{\mu}$ and $\mu = \underline{\mu}$.

In the above formulas, FB denotes a fundamental value under certainty for a particular starting dividend value. The starting dividend values can be arbitrary. A fundamental value under certainty for each starting dividend value is computed by simulating a large number of dividend realizations, discounting them and taking expectations with respect to the number of generated realizations. This is repeated for a range of starting dividend values for each of the regimes. After the vector of fundamental points has been generated, a smooth curve is fitted through them to form a fundamental value curve under certainty for each state. The choice of the fitted function as well as some of the difficulties involved in such a choice are outlined in the section on the implementation.

After fitting a curve through the fundamental values under certainty for each regime, the fundamental boundaries are given in the form of continuous functions. This is why, for simplicity, in the implementation of the algorithm the arbitrary starting dividend values are chosen to be the positive integers up to some large enough cut-off point. "Large enough" in this context depends on the parameters of the dividend generating process. For each set of parameters there will be a value, which will have a negligibly small probability of being reached or exceeded by the dividend realizations. This will then be used as a cut-off point for the starting dividends.

The reason why continuous functions are needed, rather than simply a vector of fundamental values under certainty, is because the dividend observations have a continuous state space. At any point in time the dividend value obtained by the stochastic differential equations can be any positive real number. To obtain a fundamental value at this point, the investor will take this positive real number as a starting value and discount future dividend values. Since the Monte Carlo algorithm described above tends to be quite computationally intensive, we do it only once and store the fundamental value functions under certainty. As these are continuous functions of the starting dividends, there is no problem for an investor to pick the current dividend value as a starting dividend and see what fundamental boundary point corresponds to it. Doing this for both regimes and applying the relevant conditional probabilities, the estimated true fundamental value for each risky asset is obtained.

3.4.2 Bayesian Updating and Regime Probabilities

Since each state of every industry is characterized by a unique mean-reverting dividend value, the lack of knowledge regarding the current regime is an obstacle for investors when attempting to assign a fundamental value to the risky asset. Past observations of dividend values, however, can be used to estimate the process governing the regime switches. As discussed above, the regime switching process in our formulation of the states of each industry forms a hidden Markov model. Many excellent texts exist that deal with this topic. In the following we choose to follow mainly the exposition in Elliott et al. (1995) and the related text Aggoun & Elliott (2004) because of their comprehensiveness and generality. The proofs of all secondary results are reproduced in the Appendix for the purpose of easier reference.

The term hidden Markov model is synonymous with the term partially observed stochastic dynamical system model. The simplest representative of this class of models is a model with both states and observations in a discrete set and in discrete time. The results, however, can be easily generalized to situations where the observations take values in a continuous set or where both the states and measurements take values in continuous sets. The latter case gives rise to the famous Kalman filter (see e.g. Jazwinski (1970), Fraser (2009)).

Modern treatments of the subject prefer to use reference probability methods over the semimartingale methods popular in the past. This entails a procedure in which the original estimation and control problem is reformulated by means of a probability measure change so that Fubini-type methods for identically and independently distributed random variables can be applied. The results are then translated back in terms of the original measure by means of a reverse measure change. The benefits of such an approach lie in its simplicity and generality.

Our problem consists of a two-state Markov chain, which is observed in noise in the form of many random fluctuations around the average dividend values in each state of the industry. The information that is revealed to the investors is of the form of the stochastic differential equations which specify how dividends are generated. The task is then to estimate conditional probabilities for the states of the signal based on the noisy dividend observations. We will first consider the case where both the states and the observations take values in a discrete set. The results are then generalized to the case of observations in a continuous set, which is of direct interest for our problem.

3.4.2.1 Discrete States and Discrete Observations

All processes are defined initially on a probability space (Ω, Σ, P). Consider a system whose state is described by a finite-state, homogenous Markov chain $\{X_k\}$, $k \in \mathbb{N}$ in discrete time. The initial state X_0 is assumed given or its distribution known. Suppose the state space of X_k has N elements. Then the state space can be represented without loss of generality as the set

$$S_X = \{e_1, \ldots, e_N\}, \tag{3.21}$$

where e_i are standard basis vectors in $\mathbb{R}^N$ in which the ith element of e_i is one while all the others are zero. To see this, consider a process $\{M_k\}$ having an arbitrary finite set as its state space: $S_M = \{s_1, \ldots, s_N\}$. One can use indicator functions $\phi_k(s_i)$ of the form $\phi_k(s_i) = 0$ if $i \neq k$ and $\phi_k(s_k) = 1$ to transform $\{M_k\}$ into $\{X_k\}$ by writing $X_k = (\phi_1(M_k), \ldots, \phi_N(M_k))$, so that at any time k only one component of X_k is one and the others are zero.

Denote by $\Sigma_k^0 = \sigma\{X_0, \ldots, X_k\}$ the $\sigma - algebra$ generated by $X_0, \ldots, X_k$, and write $\left\{\Sigma^k\right\}$ for the complete filtration generated by the Σ_k^0. Since the signal process follows a Markov chain, we can use the Markov property

$$P\left(X_{k+1} = e_j | \Sigma_k\right) = P\left(X_{k+1} = e_j | X_k\right).$$

Denote the matrix of transitional probabilities as follows:

$$a_{ji} = P\left(X_{k+1} = e_j | X_k = e_i\right), \quad A = (a_{ji}) \in \mathbb{R}^{N \times N}. \tag{3.22}$$

Note that since at each time k the realizations of X_k are vectors of indicator functions showing which state the system is currently in, the X_k are in fact simple random variables. For such random variables, expectations are interchangeable with probabilities:

$$E\left[\langle X_K, e_i\rangle\right] = \sum_{j=1}^{N}\langle e_j, e_i\rangle P\left(X_k = e_j\right) = P\left(X_k = e_i\right). \tag{3.23}$$

Using this property,

$$E\left[X_{k+1}|\Sigma_k\right] = E\left[X_{k+1}|X_k\right] = AX_k. \tag{3.24}$$

In other words, AX_k is the predictable part of the signal process. Using Doob's decomposition theorem (see Doob (1953)), we can give the following semi-martingale representation of the Markov chain $\{X_k\}$:

$$X_{k+1} = AX_k + V_{k+1}, \tag{3.25}$$

where $V_{k+1} := X_{k+1} - AX_k$ is a zero-mean martingale increment. To see this note that $E\left[AX_k|X_k\right] = AX_k$. Then:

$$E\left[V_{k+1}|\Sigma_k\right] = E\left[X_{k+1} - AX_k|X_k\right] = AX_k - AX_k = 0. \tag{3.26}$$

Equation (3.25) is known as the state equation.

As noted, the signal process X is not observed directly. Let $c\left(\cdot,\cdot\right)$ be a function taking values in a finite range. Then, agents actually observe the process

$$Y_{k+1} = c\left(X_k, w_{k+1}\right), \quad k \in \mathbb{N}, \tag{3.27}$$

where the terms w_k are a sequence of i.i.d random variables, independent of V_k. Denote by $\{\Xi_k\}$ the $\sigma - algebra$ generated by $X_0, \ldots, X_k$ and $Y_0, \ldots, Y_k$, while $\{\Upsilon_k\}$ will denote the complete $\sigma - algebra$ generated by $Y_0, \ldots, Y_k$. Further, suppose the range of $c\left(\cdot,\cdot\right)$ consists of M points. Then, similar to the discussion above, we can represent the state space of Y_k with a set of standard basis vectors:

$$S_Y = \left\{f_1, \ldots, f_M\right\}, \quad f_j = (0, \ldots, 1, \ldots, 0)' \in \mathbb{R}^M, \tag{3.28}$$

where the unit element is the jth element.

Equation (3.27) and the fact that V_k and w_k are mutually independent implies

$$P\left(Y_{k+1} = f_j|X_0, \ldots, X_k, Y_0, \ldots, Y_k\right) = P\left(Y_{k+1} = f_j|X_k\right).$$

Denote the matrix of conditional probabilities C as follows:

$$c_{ji} = P\left(Y_{k+1} = f_j|X_k = e_i\right), \quad C = \left(c_{ji}\right) \in \mathbb{R}^{M\times N}. \tag{3.29}$$

Since these are conditional probabilities, the column sums must add up to one: $\sum_{j=1}^{M} c_{ji} = 1$ and $c_{ji} \geq 0, 1 \leq j \leq M, 1 \leq i \leq N$.

Using this notation, as was the case with the signal process, we have:

$$E\left[Y_{k+1}|X_k\right] = CX_k. \tag{3.30}$$

Denoting by $W_{k+1} := Y_{k+1} - CX_k$ the zero-mean (P, Ξ) martingale increments, we can represent the observation process as a semimartingale as well:

$$Y_{k+1} = CX_k + W_{k+1}. \tag{3.31}$$

The fact that W_{k+1} has a zero mean is verified in exactly the same way as above. We first note the conditional expectation $E\left[CX_k|X_k\right] = CX_k$ and then we have

$$E\left[W_{k+1}|\Xi_k\right] = E\left[Y_{k+1} - CX_k|X_k\right] = CX_k - CX_k = 0.$$

Since w_k and V_k are mutually independent, W_k and V_k are also independent. Equation (3.31) is commonly referred to as an observation equation.

We introduce the following notation in order to avoid unnecessary clutter. Denote the ith element of Y_k by $Y_k^i := \langle Y_k, f_i \rangle$ so that $Y_k = \left(Y_k^1, \ldots, Y_k^M\right)', k \in \mathbb{N}$. At each time k exactly one element is equal to one, with the others being zero. That is, $\sum_{i=1}^{M} Y_k^i = 1$. Further, denote the conditional expectation of the ith element of the vector of observations by $c_{k+1}^i := E\left[Y_{k+1}^i|\Xi_k\right] = \sum_{j=1}^{N} c_{ij} \langle e_j, X_k \rangle$ and

$$c_{k+1} = \left(c_{k+1}^1, \ldots, c_{k+1}^M\right)' = E\left[Y_{k+1}|\Xi_k\right] = CX_k. \tag{3.32}$$

Without loss of generality assume that $c_k^i > 0, 1 \leq i \leq M, k \in \mathbb{N}$. Since these are conditional probabilities, we also have $\sum_{i=1}^{M} c_k^i = 1, k \in \mathbb{N}$.

The following result (see Elliott et al., 1995) will be useful in subsequent calculations.

Lemma 3.4.1 (Second Moment of V_k) *Let diag(z) denote the diagonal matrix with vector z on its diagonal. Then:*

$$\begin{aligned} V_{k+1}V_{k+1}' &= diag\left(AX_k\right) + diag\left(V_{k+1}\right) - A\ diag\left(X_k\right)A' \\ &- AX_kV_{k+1}' - V_{k+1}\left(AX_k\right)' \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} \langle V_{k+1} \rangle &:= E\left[V_{k+1}V_{k+1}'|\Sigma_k\right] \\ &= E\left[V_{k+1}V_{k+1}'|X_k\right] \\ &= diag\left(AX_k\right) - A\ diag\left(X_k\right)A'. \end{aligned} \tag{3.34}$$

Similarly, it can be shown that

$$\langle W_{k+1}\rangle := E\left[W_{k+1}W'_{k+1}|\Xi_k\right] = diag\left(CX_k\right) - C\ diag\left(X_k\right)C'.$$

To summarize, the state space signal model for a Markov chain hidden in noise and taking only discrete values under the probability measure P can be represented as follows:

$$\begin{aligned} X_{k+1} &= AX_k + V_{k+1}, \\ Y_{k+1} &= CX_k + W_{k+1}, \quad k \in \mathbb{N}, \end{aligned} \tag{3.35}$$

where $X_k \in S_X$, $Y_k \in S_Y$, A and C are matrices of transition probabilities defined in equations (3.22) and (3.29) satisfying the following conditions:

$$\sum_{j=1}^{N} a_{ji} = 1, \quad a_{ji} \geq 0, \tag{3.36}$$

$$\sum_{j=1}^{M} c_{ji} = 1, \quad c_{ji} \geq 0. \tag{3.37}$$

V_k and W_k are mutually independent martingale increments satisfying:

$$\begin{aligned} E\left[V_{k+1}|\Sigma_k\right] = 0, &\qquad E\left[W_{k+1}|\Xi_k\right] = 0, \\ \langle V_{k+1}\rangle := E\left[V_{k+1}V'_{k+1}|X_k\right] &= diag\left(AX_k\right) - A\ diag\left(X_k\right)A', \\ \langle W_{k+1}\rangle := E\left[W_{k+1}W'_{k+1}|X_k\right] &= diag\left(CX_k\right) - C\ diag\left(X_k\right)C'. \end{aligned}$$

3.4.2.2 Change of Measure

As discussed in the introduction to this section, the modern approach to handling filtering problems involves the use of reference probability methods. This entails constructing a new probability measure $\overline{P}$, under which the entire observation process $\{Y_k\}, k \in \mathbb{N}$ will be a sequence of identically and independently distributed random variables. In other words, the dependence on the signal process will be eliminated and it will no longer be possible to represent the observation process as a semimartingale. Rather, the observation process will be simply a sequence of i.i.d random variables. This will greatly simplify the analysis and once the optimal filter under these circumstances is derived, the results will simply be translated back to the original probability measure P.

Keeping the notation from the previous section, assume without loss of generality that $c_l^i > 0, 1 \leq i \leq M, l \in \mathbb{N}$. Define the values

$$\lambda_l = \prod_{i=1}^{M} \left(\frac{M^{-1}}{c_l^i}\right)^{Y_l^i}, \tag{3.38}$$

and

$$\Lambda_k = \prod_{l=1}^{k} \lambda_l. \tag{3.39}$$

An additional advantage to taking the sets of standard basis vectors $S_X = \{e_1, \ldots, e_N\}$ and $S_Y = \{f_1, \ldots, f_M\}$ as the state spaces of $\{X_k\}$ and $\{X_k\}$ respectively is the fact that any real function $f(X)$ can be expressed as a linear functional $f(X) = \langle f, X \rangle$, where $\langle f, e_i \rangle = f(e_i) = f_i$ and $f = (f_1, \ldots, f_N)$. Thus, if we denote $X^i = \langle X, e_i \rangle$, we have:

$$f(X) = \sum_{i=1}^{N} f(e_i) X^i = \sum_{i=1}^{N} f_i X^i. \tag{3.40}$$

Also note that since $Y_l^i = 1$ for only one i and $Y_l^i = 0$ otherwise, the definition of λ_l in (3.38) means that it is simply the product of $M-1$ unity terms and one non-unity term. Using this fact together with the property shown in equation (3.40) allows λ_l to be represented as $\lambda_l = \lambda_l(Y_l) = \sum_{i=1}^{M} \frac{Y_l^i}{Mc_l^i}$.

Next, we show that λ_k, as defined in equation (3.38), meets the necessary conditions to serve as the basis of the new probability measure that we want to construct (see e.g. Shreve (2004), p.210 or Aggoun & Elliott (2004), p. 136). More specifically, in order to ensure that the sequence $\{\Lambda_k\}$ is a (P, Ξ_k) martingale, we need the following result:

Lemma 3.4.2 (Conditional Expectation of λ_k) *Let λ_k be defined as in equation (3.38). Then:*

$$E[\lambda_{k+1} | \Xi_k] = 1. \tag{3.41}$$

Now that it has been proven that λ_k as defined in equation (3.38) satisfies the above condition, we can construct a new probability measure $\overline{P}$ as follows:

$$\left.\frac{d\overline{P}}{dP}\right|_{\Xi_k} = \Lambda_k, \tag{3.42}$$

where Λ_k is defined as in equation (3.39) and the notation $\left.\frac{d\overline{P}}{dP}\right|_{\Xi_k}$ denotes the restriction of the Radon-Nikodym derivative $\frac{d\overline{P}}{dP}$ to the $\sigma - algebra$ Ξ_k. The existence of $\overline{P}$ is guaranteed by Kolmogorov's Extension Theorem (see e.g. Ash & Doléans-Dade (2000), p. 118). In other words, for any Ξ_k-measurable random variable ϕ:

$$\overline{E}[\phi] = \int \phi d\overline{P} = \int \phi \frac{d\overline{P}}{dP} = \int \phi \Lambda_k dP = E[\Lambda_k \phi],$$

where $\overline{E}$ and E are expectations under $\overline{P}$ and P respectively.

When dealing with filtering problems using the reference probability method, one of the most important available tools is the conditional form of Bayes' Theorem. This result allows conditional expectations under two different measures to be related. In our case, we would like to find a relation between the expectations $\overline{E}$ and E. This relation is in the following form.

Theorem 3.4.3 (Conditional Bayes Theorem) *Let* (Ω, Σ, P) *be a probability space and* $\Xi \subset \Sigma$ *be a sub-*σ*-algebra. Further, let* $\overline{P}$ *be another probability measure that is absolutely continuous with respect to* P *and has a Radon-Nikodym derivative*

$$\frac{d\overline{P}}{dP} = \Lambda.$$

Then, if ϕ *is any integrable* Σ*-measurable random variable, we have:*

$$\overline{E}\left[\phi|\Xi\right] = \frac{E\left[\Lambda\phi|\Xi\right]}{E\left[\Lambda|\Xi\right]}.$$

Because of the importance of this result, the proof is fully reproduced in the Appendix.

The above result holds for sequences of random variables as well. A sequence of random variables $\{\phi_k\}$ is said to be Ξ-adapted if ϕ_k is Ξ_k-measurable for every k. Then, applying theorem (3.4.3) to the probability measures P and $\overline{P}$ defined in equation (3.42), we have the following version of the conditional Bayes theorem given in (3.4.3):

Lemma 3.4.4 (Conditional Bayes Theorem for Sequences of Random Variables) *If* $\{\phi_k\}$ *is a* Ξ*-adapted integrable sequence of random variables, then:*

$$\overline{E}\left[\phi_k|\Upsilon_k\right] = \frac{E\left[\Lambda_k\phi_k|\Upsilon_k\right]}{E\left[\Lambda_k|\Upsilon_k\right]}.$$

We can now turn our attention back to the definition of the new probability measure $\overline{P}$ and verify that the main goal we set out to achieve in the beginning of this section has been attained. More specifically, the observation process $\{Y_k\}$ has been transformed from a semimartingale under P into a sequence of independent and identically distributed random variables under $\overline{P}$. The following lemma summarizes this result.

Lemma 3.4.5 (Observation Process under $\overline{P}$) *Under the probability measure* $\overline{P}$ *defined in (3.42), the observation process* $\{Y_k\}$, $k \in \mathbb{N}$, *is a sequence of independent and identically distributed random variables each having a uniform distribution that assigns probability* $\frac{1}{M}$ *to each point* f_i, $1 \leq i \leq M$, *in its range space.*

As with most of the previous results, the proof is given in the Appendix.

Note, however, that even though $\overline{P}$ transformed $\{Y_k\}$ into a sequence of independent and identically distributed random variables, the signal process $\{X_k\}$ remains a Markov chain with transition matrix A under $\overline{P}$. To see this, note that using lemmas (1.4.2) and (1.4.4) we have:

$$\begin{aligned}\overline{E}\left[X_{k+1}|\Xi_k\right] &= \frac{E\left[\Lambda_{k+1}X_{k+1}|\Xi_k\right]}{E\left[\Lambda_{k+1}|\Xi_k\right]}\\ &= E\left[\lambda_{k+1}X_{k+1}|\Xi_k\right] = AX_k.\end{aligned}$$

Having transformed the observation process $\{Y_k\}$ by means of constructing a new probability measure $\overline{P}$ and having verified its new properties, the final step in the optimal estimation process when using the reference probability method is to translate results back to the original probability measure P by means of a reverse measure change. This is the main subject of the section to follow.

3.4.2.3 Reverse Measure Change

The desired outcome in this section is to start with the probability measure $\overline{P}$ defined in the previous section, such that:

1. the process $\{X_k\}$ is a finite-state Markov chain with transition matrix A and

2. the process $\{Y_k\}, k \in \mathbb{N}$, is a sequence of independent and identically distributed random variables with

$$\overline{P}\left(Y_{k+1}^j = 1|\Xi_k\right) = \overline{P}\left(Y_{k+1}^j = 1\right) = \frac{1}{M},$$

and to then translate all result back to the original measure P, under which the observation process $\{Y_k\}$ is a semimartingale.

Let $C = (c_{ji}), 1 \leq j \leq M, 1 \leq i \leq N$ be a matrix such that $c_{ji} \geq 0$ and $\sum_{j=1}^{M} c_{ji} = 1$. In this section we construct a probability measure P, such that under P the hidden Markov model described in equation (3.35) still holds and $E\left[Y_{k+1}|\Xi_k\right] = CX_k$. As in the previous section, denote the conditional expectation of the observation process by:

$$c_{k+1} = E\left[Y_{k+1}|\Xi_k\right] = CX_k,$$

with element i being equal to $c_{k+1}^i = \langle c_{k+1}, f_i\rangle = \langle CX_k, f_i\rangle$, so that:

$$\sum_{i=1}^{M} c_{k+1}^i = 1. \tag{3.43}$$

The construction of P from $\overline{P}$ is inverse to that of $\overline{P}$ from P, which was illustrated in the previous section. Define:

$$\overline{\lambda}_l = \prod_{i=1}^{M} \left(Mc_l^i\right)^{Y_l^i}, \quad l \in \mathbb{N}, \tag{3.44}$$

and

$$\overline{\Lambda}_k = \prod_{l=1}^{k} \overline{\lambda}_l. \tag{3.45}$$

Note that unlike in the previous section, since we no longer divide by c_k^i in the construction of P, we do not need to impose the additional requirement that c_k^i must be strictly positive.

As with the construction of $\overline{P}$ in the previous chapter, we again need to make sure that the $\overline{\lambda}_k$ meet the necessary conditional expectation conditions in order to serve as building blocks for the Radon-Nikodym derivative that will be defined below. The result is analogous to what has already been proved for λ_k.

Lemma 3.4.6 (Conditional Expectation of $\overline{\lambda}_k$) *With c_k^i and $\overline{\lambda}_k$ defined as in equations (3.43) and (3.44) respectively, we have the following result:*

$$\overline{E}\left[\overline{\lambda}_{k+1} | \Xi_k\right] = 1. \tag{3.46}$$

The proof follows closely the proof of lemma (1.4.2).

Similar to the way that $\overline{P}$ was specified in the previous section, we define the new probability measure P by setting the restriction of the Radon-Nikodym derivative $\frac{dP}{d\overline{P}}$ to the σ-algebra Ξ_k equal to $\overline{\Lambda}_k$:

$$\left.\frac{dP}{d\overline{P}}\right|_{\Xi_k} = \overline{\Lambda}_k, \tag{3.47}$$

where $\overline{\Lambda}_k$ is defined as in equation (3.45). As was the case with $\overline{P}$, the existence of P is guaranteed by Kolmogorov's Extension Theorem (Ash & Doléans-Dade (2000)).

The last issue that needs to be tackled is to verify that the new probability measure P indeed transforms the sequence of independent and identically distributed random variables $\{Y_k\}$ back into a semimartingale with a predictable part CX_k. The following lemma formalizes this result.

Lemma 3.4.7 (Observation Process under P) *Under the probability measure P, as defined in equation (3.47),*

$$E\left[Y_{k+1} | \Xi_k\right] = CX_k.$$

The proof is similar to that of lemma (1.4.5) and is provided in the Appendix.

We have now demonstrated the relationship between conditional expectations under two different probability measures, formalized by the conditional Bayes theorem (3.4.3), as well as the construction of the two specific probability measures necessary in order to solve the filtering problem using the reference probability method. The next section uses these building blocks in order to compute unnormalized estimates and ultimately, conditional probabilities for the state of the signal process.

3.4.2.4 Unnormalized Estimates and Conditional Probabilities

We are now in a position to calculate unnormalized estimates of the conditional probabilities of the signal states for the case of discrete time signal processes with discrete states and discrete observations. The same notation as in the previous chapters will be used. Namely, we denote by Υ_k the complete σ-algebra generated by observations of $Y_1, \ldots, Y_k$ and by Ξ_k the complete σ-algebra generated by observing $X_0, X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$. We assume there exists a probability measure $\overline{P}$ such that, under $\overline{P}$, $X_{k+1} = AX_k + V_{k+1}$, where V_k is a zero-mean $\left(\overline{P}, \Xi_k\right)$ martingale increment: i.e. $\overline{E}\left[V_{k+1}|\Xi_k\right] = 0$. Furthermore, under $\overline{P}$, the observation process is assumed to be a sequence of independent and identically distributed random variables, each distributed uniformly over the M elements in its range space: i.e. $\overline{P}\left(Y_k^j = 1\right) = \frac{1}{M}$. Each Y_k is also assumed mutually independent of the corresponding V_k.

In the previous sections it was already shown that $\overline{E}\left[V_{k+1}|\Xi_k\right] = 0$. Equation (3.26) stated this under P, however, as we noted in the section on the change of measure, under $\overline{P}$ the properties of $\{X_k\}$ are preserved. Therefore, any result for $\{X_k\}$ under P applies to $\overline{P}$ as well. However, more can be said about the conditional expectation of V_k.

Since V_k and Y_k are mutually independent, and since conditioning on both Ξ_k and Υ_{k+1} brings more information than conditioning on Υ_{k+1} only ($\Upsilon_{k+1} \subset \Xi_k \cup \Upsilon_{k+1}$), using the double expectations property we can observe that:

$$\begin{aligned} \overline{E}\left[V_{k+1}|\Upsilon_{k+1}\right] &= \overline{E}\left[\overline{E}\left[V_{k+1}|\Xi_k, \Upsilon_{k+1}\right]|\Upsilon_{k+1}\right] \\ &= \overline{E}\left[\overline{E}\left[V_{k+1}|\Xi_k\right]|\Upsilon_{k+1}\right] = 0. \end{aligned} \tag{3.48}$$

Based on these assumptions, the probability measure P is defined as in equation (3.47). Since it is the conditional probability of the signal states under P the we are ultimately interested in, we would also need the conditional Bayes theorem for sequences of random variables. Using lemma (1.4.4), for any

Ξ-adapted sequence of random variables $\{\phi_k\}$, we have:

$$E\left[\phi_k|\Upsilon_k\right] = \frac{\overline{E}\left[\Lambda_k \phi_k|\Upsilon_k\right]}{\overline{E}\left[\Lambda_k|\Upsilon_k\right]} \tag{3.49}$$

The quantity $\overline{E}\left[\Lambda_k \phi_k|\Upsilon_k\right]$ in the numerator is what we referred to as the unnormalized conditional expectation of the random variable ϕ_k based on information about the observation process $\{Y_k\}$.

The reason for the above terminology may not be immediately obvious. In order to simplify notation, define $q_k\left(e_r\right), 1 \le r \le N, k \in \mathbb{N}$ to be the unnormalized conditional probability of the signal being in state r at time k given the history of observations up until and including time k. That is:

$$\overline{E}\left[\Lambda_k \langle X_k, e_r\rangle|\Upsilon_k\right] = q_k\left(e_r\right).$$

Since $\sum_{i=1}^{N}\langle X_k, e_i\rangle = 1$, we have:

$$\sum_{i=1}^{N} q_k\left(e_i\right) = \overline{E}\left[\Lambda_k \sum_{i=1}^{N}\langle X_k, e_i\rangle \middle| \Upsilon_k\right] = \overline{E}\left[\Lambda_k|\Upsilon_k\right].$$

In other words, the denominator in equation (3.49) normalizes each $q_k\left(e_r\right)$ by dividing it by the sum over all possible states of the signal X_k.

Therefore, using the above fact and the result in equation (3.49), we obtain the normalized conditional probability of the signal X_k being in state r at time k as:

$$\begin{aligned} p_k\left(e_r\right) &= E\left[\langle X_k, e_r\rangle|\Upsilon_k\right] \\ &= \frac{q_k\left(e_r\right)}{\sum_{j=1}^{N} q_k\left(e_j\right)}. \end{aligned}$$

Although this is a conceptually correct formulation of the quantity we are interested in, the representation given above is not specific enough to be implemented straight away. For computational purposes, it would be preferable to obtain a recursive relationship, which would allow q_k to be computed from q_{k-1}. Defining $c_j\left(Y_k\right) = M\prod_{i=1}^{M} c_{ij}^{Y_k^i}$, and writing any N-dimensional vector v as $v_{(\cdot)} = \left(v_1, \ldots, v_N\right)'$, the desired recursive expression for the vector of unnormalized conditional expectations q_k is formalized in the following theorem.

Theorem 3.4.8 (Recursive Filter) *For $k \in \mathbb{N}$ and $1 \le r \le N$, the recursive filter for the unnormalized estimates of the states of the signal process $\{X_k\}$ is given by:*

$$q_{k+1} = A\ diag\left(q_k\right) c_{(\cdot)}\left(Y_{k+1}\right), \tag{3.50}$$

where A is a matrix of transition probabilities, $diag\left(q_k\right)$ is a diagonal matrix with a main diagonal q_k, and $c_{(\cdot)}\left(Y_{k+1}\right)$ is a vector of probabilities of observing the realized value of the observation process $\{Y_{k+1}\}$ on condition that the signal was in a particular state at time k.

As usual, the proof is provided in the Appendix.

Note that theorem (3.4.8) gives a recursive expression for the unnormalized estimates of the signal states in the form of a vector. For each state of the signal process, the corresponding element of this vector is given by:

$$\begin{aligned} q_{k+1}\left(e_r\right) &= \overline{E}\left[\overline{\Lambda}_{k+1}\langle X_{k+1}, e_r\rangle | \Upsilon_{k+1}\right] \\ &= \sum_{j=1}^{N} q_k\left(e_j\right) a_{rj} M \prod_{i=1}^{M} c_{ij}^{Y_{k+1}^i}. \end{aligned}$$

The intuition behind the above equation is that the unnormalized estimate for the conditional probability of the signal process being in state r at time $k+1$ is given by the sum over all possible states of the unnormalized estimate for the previous period k, multiplied by the transitional probability of moving from each state to state r, multiplied by the probability of observing the realized value Y_{k+1}^i on condition that the signal process in the previous period X_k was in each of its possible states j. Also notice that the recursive relation (3.50) is linear in q.

3.4.2.5 A General Unnormalized Recursive Filter

The recursive filter for the unnormalized estimates of the signal states given in equation (3.50) above was the solution to the particular estimation problem we were facing – namely, the estimation of conditional probabilities for the two states of each industry in our model based on observations of the dividend realizations and knowledge of the underlying dividend generating process. It is possible to reformulate this recursive expression so that it is capable of handling much more general signal processes than our particular task required. In this section we briefly outline the derivation of a general unnormalized recursive filter. We also show that the expression obtained in equation 3.50 is a special case of this more general result.

We will continue to work under the probability measure $\overline{P}$, such that:

$$X_{k+1} = AX_k + V_{k+1}, \tag{3.51}$$

and the observation process $\{Y_k\}$ is a sequence of independent and identically distributed random variables, each with a uniform distribution over the M elements of its range space $f_1, \ldots, f_M$. In order to reach the goal of greater generality, it would be helpful to introduce the following new notation. Let $\{H_k\}, k \in \mathbb{N}$ be any integrable sequence of random variables. We shall write:

$$\gamma_k(H_k) = \overline{E}\left[\overline{\Lambda}_k H_k | \Upsilon_k.\right] \tag{3.52}$$

The process H can be both scalar- or vector-valued.

An application of lemma (1.4.4) to the random variable H_k yields:

$$E[H_k|\Upsilon_k] = \frac{\overline{E}\left[\overline{\Lambda}_k H_k|\Upsilon_k\right]}{\overline{E}\left[\overline{\Lambda}_k|\Upsilon_k\right]} = \frac{\gamma_k(H_k)}{\gamma_k(1)}. \tag{3.53}$$

Therefore, $\gamma_k(H_k)$ can be interpreted as the unnormalized conditional expectation of H_k given the information set Υ_k. Again we shall assume that initial values for the recursions are either given or their distribution is known.

Without loss of generality assume $\{H_k\}, k \in \mathbb{N}$ is an integrable scalar sequence of random variables. Write $\Delta H_{k+1} = H_{k+1} - H_k$, i.e. $H_{k+1} = H_k + \Delta H_{k+1}$. Then we have the following representation:

$$\gamma_{k+1}(H_{k+1}) = \overline{E}\left[\overline{\Lambda}_{k+1} H_k|\Upsilon_{k+1}\right] + \overline{E}\left[\overline{\Lambda}_{k+1}\Delta H_{k+1}|\Upsilon_{k+1}\right].$$

Concentrating on the first term on the right-hand side of the equation and using properties (3.23) and (3.40), we have:

$$\begin{aligned}
\overline{E}\left[\overline{\Lambda}_{k+1} H_k|\Upsilon_{k+1}\right] &= \overline{E}\left[\overline{\Lambda}_k H_k \overline{\lambda}_{k+1}|\Upsilon_{k+1}\right] \\
&= \overline{E}\left[\overline{\Lambda}_k H_k M \prod_{i=1}^{M} \left(\langle CX_k, f_i\rangle\right)^{Y_{k+1}^i} \middle| \Upsilon_{k+1}\right] \\
&= \sum_{j=1}^{N} \overline{E}\left[\overline{\Lambda}_k H_k \langle X_k, e_j\rangle|\Upsilon_k\right] M \prod_{i=1}^{M} c_{ij}^{Y_{k+1}^i} \\
&= \sum_{j=1}^{N} c_j(Y_{k+1}) \langle \gamma_k(H_k X_k), e_j\rangle.
\end{aligned}$$

The second equality above comes from a simple substitution using the definition of $\overline{\lambda}_{k+1}$ as shown in equation (3.44). The third equality follows from a number of facts similar to those used in the proof of theorem (3.4.8). Firstly, $\overline{\Lambda}_k H_k$ is replaced by $\sum_{j=1}^{N} \overline{\Lambda}_k H_k \langle X_k, e_j\rangle$. This is merely another representation of the same quantity since $\sum_{j=1}^{N}\langle X_k, e_j\rangle = 1$. Changing the order of the summation and the expectation operators is allowed by Fubini's theorem. Secondly, since $\{Y_k\}$ are i.i.d. random variables under $\overline{P}$, conditioning on Υ_{k+1} instead of Υ_k does not bring any new useful information and therefore the conditional distribution in the third equality remains the same using any of these two conditioning sets.

Furthermore, using property (3.23) the expectation $\overline{E}\left[M\prod_{i=1}^{M}\left(\langle CX_k, f_i\rangle\right)^{Y_{k+1}^i}\right]$ is the same as the probability $\overline{P}\left[M\prod_{i=1}^{M}\left(\langle CX_k, f_i\rangle\right)^{Y_{k+1}^i}\right]$. However, instead of multiplying the transitional probability matrix C times vector X_k and then taking its i-th element, the transitional probabilities from each state j to state i are summed across all possible states N of X_k. In other words, the conditional probability of the observation process $\prod_{i=1}^{M}\left(\langle CX_k, f_i\rangle\right)^{Y_{k+1}^i}$ is written as the sum of the probabilities of observing the realized value of Y_{k+1} on condition that X_k was in each of its possible N states. The last equality follows from using the definitions of $c_j\left(Y_{k+1}\right)$ and $\gamma\left(\cdot\right)$ introduced in theorem (3.4.8) and equation (3.52) respectively.

Note that unlike in theorem (3.4.8), where $q_{k+1}\left(e_r\right) = \overline{E}\left[\overline{\Lambda}_{k+1}\langle X_{k+1}, e_r\rangle|\Upsilon_{k+1}\right]$ denoted the conditional expectation of an element of X_{k+1}, in equation (3.52) H_k is a general scalar- or vector-valued random variable. This brings about a problem when trying to find a recursive formulation for $\gamma_{k+1}\left(H_{k+1}\right)$. A difficulty arises because the estimate for $\gamma_{k+1}\left(H_{k+1}\right)$ introduces a new term – $\gamma_k\left(H_kX_k\right)$ – which includes the additional random variable X_k, so the expression for $\gamma_{k+1}\left(H_{k+1}\right)$ is no longer recursive in H only.

As mentioned in Elliott et al. (1995), this can be overcome by taking advantage of the structure of X_{k+1}. By noting that X_{k+1} is a vector whose every element but one is zero with the remaining element being equal to one, and concentrating on the recursive representation of $\gamma_{k+1}\left(H_{k+1}X_{k+1}\right)$ instead of $\gamma_{k+1}\left(H_{k+1}\right)$, it can be shown by an argument similar to the one above that the recursive representation of $\gamma_{k+1}\left(H_{k+1}X_{k+1}\right)$ introduces the term $\gamma_k\left(H_kX_kX_k'\right)$. Writing this in terms of standard basis vectors, we have $\sum_{i=1}^{N}\langle\gamma_k\left(H_kX_k\right), e_i\rangle e_ie_i'$. It is evident from this representation that the estimate for $\gamma_{k+1}\left(H_{k+1}X_{k+1}\right)$ can be recursively expressed in terms $\gamma_k\left(H_kX_k\right)$.

In order to eliminate the additional X_{k+1}, we make the following observation. Denote by $\underline{1}$ the vector whose every element is one: $(1, 1, \ldots, 1)' \in \mathbb{R}^N$. Then $\langle X_k, \underline{1}\rangle = \sum_{i=1}^{N}\langle X_k, e_i\rangle = 1$. This means that we can represent $\gamma_k\left(H_k\right)$ by:

$$\langle\gamma_k\left(H_kX_k\right), \underline{1}\rangle = \gamma_k\left(H_k\langle X_k, \underline{1}\rangle\right) = \gamma_k\left(H_k\right). \tag{3.54}$$

In other words, the unnormalized estimate $\gamma_k\left(H_k\right)$ is obtained by summing all the elements of $\gamma_k\left(H_kX_k\right)$. In this way, by providing a recursive representation of $\gamma_{k+1}\left(H_{k+1}X_{k+1}\right)$ and then summing both sides over all its elements, the problem of the appearance of the additional X_k terms can be circumvented.

Turning our attention to the denominator in equation (3.53), we can substitute $H_k = 1$ in equation (3.54) to obtain:

$$\gamma_k\left(1\right) = \langle\gamma_k\left(X_k\right), \underline{1}\rangle = \overline{E}\left[\overline{\Lambda}_k|\Upsilon_k\right] = \sum_{i=1}^{N} q_k\left(e_i\right)$$

using the notation in the previous section. Thus, the normalizing term $\gamma_k(1)$ in equation (3.53) can be obtained by simply summing all the elements of $\gamma_k(X_k)$, which is the same as the normalizing factor introduced in the previous section.

We wrap up this section by further specifying the general process H, while still retaining greater generality than the special case shown in the previous section. Using the same time index $k \geq 1$, assume H_k is a scalar-valued process of the form:

$$\begin{aligned} H_{k+1} &= \sum_{l=1}^{k+1} \left(\alpha_l + \langle \beta_l, V_l \rangle + \langle \delta_l, Y_l \rangle\right) \\ &= H_k + \alpha_{k+1} + \langle \beta_{k+1}, V_{k+1} \rangle + \langle \delta_{k+1}, Y_{k+1} \rangle, \end{aligned} \tag{3.55}$$

where $V_l = X_l - AX_{l-1}$ and $\alpha_l, \beta_l, \delta_l$ are Ξ-predictable processes of appropriate dimensions, i.e. $\alpha_l, \beta_l, \delta_l$ are Ξ_{l-1}-measurable, α_l is scalar, β_l is N-dimensional, and δ_l is M-dimensional. We simplify our notation in the following theorem by writing for any Ξ-adapted process $\phi_k, k \in \mathbb{N}$:

$$\gamma_{m,k}(\phi_m) = \overline{E}\left[\Lambda_k \phi_m X_k \,|\Upsilon_k\right]. \tag{3.56}$$

Theorem 3.4.9 (General Recursive Filter) *For $1 \leq j \leq M$ write $c_j = Ce_j = (c_{1j}, \ldots, c_{Mj})'$ for the jth column of $C = (c_{ij})$ and $a_j = Ae_j = (a_{1j}, \ldots, a_{Nj})'$ for the jth column of $A = (a_{ij})$. Then, the following result holds:*

$$\begin{aligned} \gamma_{k+1,k+1}(H_{k+1}) &= \sum_{j=1}^{N} c_j(Y_{k+1}) \{\langle \gamma_{k,k}(H_k) + \gamma_{k+1,k}(\alpha_{k+1} + \langle \beta_{k+1}, Y_{k+1} \rangle), e_j \rangle a_j \\ &+ \left[diag(a_j) - a_j a_j'\right] \overline{E}\left[\langle \Lambda_k X_k, e_j \rangle \beta_{k+1} \,|\Upsilon_{k+1}\right] \Big\}. \end{aligned} \tag{3.57}$$

A proof following Elliott et al. (1995) is provided in the Appendix.

The estimator of the state which was derived in equation (3.50) is simply a special case of the representation in equation (3.57). To see this, take $H_{k+1} = H_0 = 1, \alpha_l = 0, \beta_l = 0$, and $\delta_l = 0$. Applying theorem (3.4.9) we obtain the unnormalized recursive filter in equation (3.50) for $q_k = (q_k(e_1), \ldots, q_k(e_N))$ in vector form:

$$\gamma_{k+1,k+1} = q_{k+1} = \sum_{j=1}^{N} c_j(Y_{k+1}) \langle q_k, e_j \rangle a_j. \tag{3.58}$$

The normalized conditional probability of the state is then:

$$p_k = q_k \langle q_k, \underline{1} \rangle^{-1}. \tag{3.59}$$

3.4.2.6 Discrete States and Continuous-Range Observations

Although quite general in nature, the results in the previous section can only be applied under a very restricted set of circumstances. Namely, we considered problems, in which both the hidden signal process and the observation process could take values in a discrete set. In practical applications this proves quite restrictive. For instance, our specification of the observed dividend process in equation (3.14) used a mean-reverting square root process, while in equation (3.17) this was accomplished by means of a geometric Ornstein-Uhlenbeck process.

The common feature of these two dividend specifications is the modeling of the uncertainty in the economy by means of the Wiener process $W(t)$. Its independent and normally distributed increments imply that at any point in time the process can take an uncountably infinite number of values. Consequently, the model with a discrete observation space presented in the previous sections cannot be used to solve the problem of estimating conditional probabilities of the two states of each industry on the basis of the observed dividend realizations, when the dividend processes are specified as in equations (3.14) and (3.17).

These deficiencies can be remedied by using a model with discrete states but continuous-range observations. A popular way to do this is to consider a discrete-time, finite-state Markov chain, which is observed through a real- or vector-valued function subjected to noise (see Elliott et al., (1995)). For our particular problem, considering Gaussian noise is sufficient, however, considering observations with "colored noise", i.e. noise terms that are correlated across time, is also possible.

The structure of the following sections follows the sequence laid out by the presentation of the model with discrete observations. The main analytical tool is again a discrete time version of Girsanov's theorem and change of measure. In a similar fashion to the discrete observations case, a new probability measure $\overline{P}$ will be constructed, however, in this case the components of the observation process will be $N(0,1)$ i.i.d. random variables under $\overline{P}$. Generally, the extension of the results in the discrete observations case to a setting with continuous observations is relatively straightforward. Most of the results will have equivalents in continuous observations. Some modifications will be necessary in order to accommodate the different specification of the observation process. In the presentation below, we continue following Elliott et al. ((1995)).

3.4.2.7 Specification of the State and Observation Processes

As before, all processes are defined on a complete probability space (Ω, Σ, P). The index parameter k takes values in $\mathbb{N}$. Denote by $\{X_k, k \in \mathbb{N}\}$ the finite-state Markov chain representing the signal process. Using the same correspondence between a general state space and a set of unit vectors, we represent the state space of X by the set of unit vectors:

$$S_X = \{e_1, e_2, \ldots, e_N\}, \quad e_i = (0, \ldots, 0, 1, 0, \ldots, 0)' \in \mathbb{R}^N.$$

The initial state X_0 is assumed given or its distribution known. Similarly to the previous sections, X is assumed to be a homogeneous Markov chain:

$$P\left(X_{k+1} = e_j | \Sigma_k\right) = P\left(X_{k+1} = e_j | X_k\right).$$

Assume X is not observed directly, but rather through an observation process $\{y_k, k \in \mathbb{N}\}$. Also, in order to simplify calculations, let y be scalar-valued. Under P, the state space equations of the discrete state, continuous observation hidden Markov model are as follows:

$$\begin{aligned} X_{k+1} &= AX_k + V_{k+1}, \\ y_{k+1} &= c\left(X_k\right) + \sigma\left(X_k\right) w_{k+1}. \end{aligned} \tag{3.60}$$

In the above, assume y is a real-valued process. $\{w_k\}$ is a sequence of zero mean, unit variance normally distributed $N(0,1)$ i.i.d. random variables. Because $X_k \in S_X$ is an N-dimensional vector, the functions $c(\cdot)$ and $\sigma(\cdot)$ are determined by corresponding vectors $c = (c_1, c_2, \ldots, c_N)'$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)'$ in $\mathbb{R}^N$. Since we designated y to be a scalar, what we mean by the notation $c(X_k)$ and $\sigma(X_k)$ is $\langle c, X_k \rangle$ and $\langle \sigma, X_k \rangle$ respectively, where $\langle\ ,\ \rangle$ denotes the inner product in $\mathbb{R}^N$.

It is assumed $\sigma_i \neq 0$, so without loss of generality we can assume $\sigma_i > 0$, $1 \leq i \leq N$. As in the previous sections, we shall denote by $\{\Sigma_k\}, k \in \mathbb{N}$ the complete filtration generated by X. Likewise, $\{\Upsilon_k\}, k \in \mathbb{N}$ shall denote the complete filtration generated by y and $\{\Xi_k\}, k \in \mathbb{N}$ – the complete filtration generated by both X and y.

3.4.2.8 Conditional Expectations from First Principles

In this section we take a brief excursion and try to solve the problem we have directly by deriving the conditional distribution of X_k given Υ_k from first principles only, without resorting to the reference probability method. While this turns out to be possible, it will become evident why the reference probability method is still computationally and methodologically superior.

First note that since $w_k, k \in \mathbb{N}$ are $N(0,1)$ i.i.d. random variables, w_k is independent of Ξ_k and hence of $\Upsilon_k \subset \Xi_k$.

For any $t \in \mathbb{R}$ we have the conditional distribution:

$$P\left(y_{k+1} \leq t \,|\Upsilon_k\right) = \sum_{i=1}^{N} P\left(\sigma_i w_{k+1} \leq t - c_i\right) P\left(X_k = e_i \,|\Upsilon_k\right).$$

We introduce the following notation: denote the conditional expectation of the signal process by $\hat{X}_k = E\left[X_k \,|\Upsilon_k\right]$ and write $\phi_i(x) = (2\pi\sigma_i)^{-\frac{1}{2}} exp\left(\frac{-x^2}{2\sigma_i^2}\right)$ for the $N(0,\sigma_i)$ density. Then, the above conditional distribution can be rewritten as follows:

$$P\left(y_{k+1} \leq t \,|\Upsilon_k\right) = \sum_{i=1}^{N} \langle \hat{X}_k, e_i \rangle \int_{-\infty}^{t-c_i} \phi_i(x)\, dx.$$

This follows from the fact that X_k is a simple random variable and hence probabilities and expectations are interchangeable. The probability of X_k being in state j, therefore, is the same as the j-th element of the conditional expectation of X_k (see property (3.23)). Since the probability at the left tail of the distribution at $-\infty$ is zero, the conditional density of y_{k+1} given Υ_k is:

$$\sum_{j=1}^{N} \langle \hat{X}_k, e_j \rangle \phi_j\left(t - c_j\right).$$

Also, the joint distribution of the signal and observation processes is given by:

$$\begin{aligned} P\left(X_k = e_i, y_{k+1} \leq t \,|\Upsilon_k\right) &= P\left(X_k = e_i \,|\Upsilon_k\right) P\left(\sigma_i w_{k+1} \leq t - c_i\right) \\ &= \langle \hat{X}_k, e_i \rangle \int_{-\infty}^{t-c_i} \phi_i(x)\, dx. \end{aligned}$$

Then, a straightforward application of Bayes' rule yields:

$$\begin{aligned} E\left[\langle X_k, e_i \rangle \,|\Upsilon_{k+1}\right] &= P\left(X_k = e_i \,|y_{k+1}, \Upsilon_k\right) \\ &= \frac{\langle \hat{X}_k, e_i \rangle \phi_i\left(y_{k+1} - c_i\right)}{\sum_{j=1}^{N} \langle \hat{X}_k, e_j \rangle \phi_j\left(y_{k+1} - c_j\right)}. \end{aligned} \tag{3.61}$$

Note that we now have $\phi_j(y_{k+1} - c_j)$ instead of $\phi_j(t - c_j)$ because we know that the observation y_{k+1} has occurred. The numerator is then simply the joint probability of the signal process being in state i and obtaining a value for the observation process at least as large as the quantity we have observed – y_{k+1}. Unlike the case with discrete observations, here we cannot simply use the probability that the observation process is exactly equal to y_{k+1} since $y_{k+1} \in \mathbb{R}$ and hence the probability of it being exactly equal to any one value is zero.

Equation (3.61) above expressed the conditional probability of the signal process being in one of its possible states i, or equivalently, the conditional expectation of a single element of the vector X_k. Therefore, it is not difficult to obtain the conditional probability of the entire vector X_k by multiplying the above expression by all the unit vectors corresponding to each of the states of X_k and summing them:

$$\begin{aligned} E\left[X_k \left| \Upsilon_{k+1}\right.\right] &= \sum_{i=1}^{N} E\left[\langle X_k, e_i\rangle \left| \Upsilon_{k+1}\right.\right] e_i \\ &= \frac{\sum_{i=1}^{N} \langle \hat{X}_k, e_i\rangle \phi_i\left(y_{k+1} - c_i\right) e_i}{\sum_{j=1}^{N} \langle \hat{X}_k, e_j\rangle \phi_j\left(y_{k+1} - c_j\right)}. \end{aligned} \tag{3.62}$$

Finally, we can express this conditional expectation as a recursive filter for $\hat{X}_k$ as is demonstrated in the following theorem:

Theorem 3.4.10 (Recursive Filter for $\hat{X}_k$)

$$\hat{X}_{k+1} = E\left[X_{k+1} \left| \Upsilon_{k+1}\right.\right] = \frac{\sum_{i=1}^{N} \langle \hat{X}_k, e_i\rangle \phi_i\left(y_{k+1} - c_i\right) A e_i}{\sum_{j=1}^{N} \langle \hat{X}_k, e_j\rangle \phi_j\left(y_{k+1} - c_j\right)}. \tag{3.63}$$

A short proof is provided in the Appendix. From this representation, it can be seen right away what the main problem with this approach is. Equation (3.63) clearly indicates that the recursive relation for $\hat{X}_{k+1}$ is not linear in $\hat{X}_k$. Therefore, in order to derive a linear recursive filter, in the following sections we utilize the reference probability approach in a similar way to the method that was shown in the discrete observations case.

3.4.2.9 Change of Measure for Continuous-Range Observations

Let $w(\cdot)$ be a real random variable with density $\phi(w)$, and let c and σ be known constants. Write $y(\cdot) = c + \sigma w(\cdot)$ for the observation process.

The general idea is the same as in the discrete observations case – a new probability measure $\overline{P}$ is introduced, such that under $\overline{P}$, not the noise term $w(\cdot)$, but the whole observation process $y(\cdot)$ has density ϕ. This is accomplished by introducing a density λ, such that $\frac{d\overline{P}}{dP} = \lambda$. In other words:

$$\overline{P}(y \leq t) = \int_{-\infty}^{t} \phi(y)\, dy \tag{3.64}$$

$$\begin{aligned} &= \int_{\Omega} I_{y \leq t}\; d\overline{P} \\ &= \int_{\Omega} I_{y \leq t} \lambda\; dP \\ &= \int_{-\infty}^{+\infty} I_{w \leq \frac{t-c}{\sigma}} \lambda(w)\, \phi(w)\;\; dw \\ &= \int_{-\infty}^{t} \lambda(w)\, \phi(w)\;\; \frac{dy}{\sigma}. \end{aligned} \tag{3.65}$$

The last equality holds since $y(\cdot) = c + \sigma w(\cdot)$ and hence $\frac{dy}{dw} = \sigma$, so the change of the limits of integration is valid. Therefore, using equations (3.64) and (3.65), $\phi(y)$ and $\lambda(w)\frac{\phi(w)}{\sigma}$ are equivalent, and therefore it must be that:

$$\lambda(w) = \frac{\sigma\phi(y)}{\phi(w)}.$$

As was discussed in the section on the specification of the model with continuous-range observations, on the probability space (Ω, Σ, P), the observation process $\{y_k\}, k \in \mathbb{N}$ has the form $y_{k+1} = \langle c, X_k\rangle + \langle\sigma, X_k\rangle w_{k+1}$, with $w_k \sim N(0,1)$ and independent. Denote the $N(0,1)$ density by $\phi(\cdot)$. Then, the density used to specify the probability measure $\overline{P}$ becomes:

$$\begin{aligned}\lambda_l &= \frac{\langle\sigma, X_l\rangle\phi(y_l)}{\phi(w_l)}, \quad l \in \mathbb{N}, \\ \Lambda_0 &= 1,\end{aligned} \tag{3.66}$$

and

$$\Lambda_k = \prod_{l=1}^{k} \lambda_l, \quad k \geq 1.$$

Here l and k refer to time indices. The numerator in equation (3.66) is a scalar. At each time l it will correspond to the element of X_l which is equal to one. As we did in the section on the discrete observations case, we define the new probability measure $\overline{P}$ by setting the restriction of the Radon-Nikodym derivative to Ξ_k equal to Λ_k: $\left.\frac{d\overline{P}}{dP}\right|_{\Xi_k} = \Lambda_k$. Again, the existence of $\overline{P}$ is guaranteed by Kolmogorov's Extension Theorem.

The following lemma confirms that $\overline{P}$ is specified in a way, which is consistent with the motivation for its introduction.

Lemma 3.4.11 (Distribution of y_k under $\overline{P}$) *Under $\overline{P}$ the observation process $\{y_k\}$ is a sequence of $N(0,1)$ i.i.d. random variables.*

As usual, the proof is to be found in the Appendix.

Similarly to what we did in the section on change of measure for the discrete observations case, the last thing that needs to be checked is that the new probability measure $\overline{P}$, specified above, changes the distribution of the observations y_k only, while preserving the distribution of the underlying signal process X_k. This is formalized in the following lemma:

Lemma 3.4.12 (Distribution of X_k under $\overline{P}$) *Under both probability measures P and $\overline{P}$, the signal process X is a Markov process, with a transition matrix A and initial distribution p_0.*

A proof is provided in the Appendix. Albeit longer, it follows the same logic as its discrete observations equivalent. Note that we did not provide a separate lemma for the above result in the section dealing with discrete observations. This was because the result followed as a natural consequence of previously proven theorems. In this section, we did not provide separate theorems for all those ancillary results. Rather, they will be included in the proof of lemma (3.4.12).

Having defined the specification of the new probability measure $\overline{P}$, and having established its properties, the next step, when using the reference probability method, is the translation of results back to the original probability measure P. This procedure bears many similarities with the discrete observations case discussed above, and hence the discussion on the reverse change of measure presented below will be relatively brief, focusing mainly on the important distinctions between the two cases.

3.4.2.10 Reverse Change of Measure for Continuous-Range Observations

Having shown how the filtering task we are interested in can be greatly simplified by introducing the new probability measure $\overline{P}$, and thus making the distribution of the observation process $\{y_k\}$ much more analytically tractable, we proceed by going back to the original probability measure, under which $y_k = \langle c, X_k \rangle + \langle \sigma, X_k \rangle w_k$, where $w_k \sim N(0,1)$ and are also i.i.d. random variables.

To this end, suppose we start with a probability measure $\overline{P}$ on (Ω, Σ), such that under $\overline{P}$ the following statements are true:

1. $\{X_k\}, k \in \mathbb{N}$, is a Markov chain with a matrix of transition probabilities A, such that $X_{k+1} = AX_k + V_{k+1}$, where $\overline{E}[V_{k+1} | \Sigma_k] = 0$, and
2. $\{y_k\}, k \in \mathbb{N}$, is a sequence of $N(0,1)$ i.i.d. random variables (in particular, independent of X_k).

In the previous section, it was shown that such a probability measure exists, and its construction starting from P was discussed in detail. We now wish to do the opposite and specify how a probability measure P can be constructed, starting from $\overline{P}$, such that under P the underlying signal process will retain its distribution, but the observation process will be translated back to its original definition. That is, under P, we wish to have:

$$w_{k+1} := \frac{y_{k+1} - \langle c, X_k \rangle}{\langle \sigma, X_k \rangle}, \quad k \in \mathbb{N},$$

where $\{w_k\}$ is a sequence of $N(0,1)$ i.i.d. random variables. This definition is equivalent to the original formulation of the observation process with continuous-range observations: $y_{k+1} = \langle c, X_k\rangle + \langle \sigma, X_k\rangle w_{k+1}$.

In order to construct P from $\overline{P}$ we introduce the inverses of λ_l and Λ_k:

$$\begin{aligned}\overline{\lambda}_l &= \lambda_l^{-1} = \frac{\phi(w_l)}{\langle \sigma, X_{l-1}\rangle \phi(y_l)}, \quad l \in \mathbb{N}, \\ &= \overline{\Lambda}_0 = 1,\end{aligned}$$

and

$$\overline{\Lambda}_k = \prod_{l=1}^{k} \overline{\lambda}_l, \quad k \geq 1.$$

Define P by setting the restriction of the Radon-Nikodym derivative to Ξ_k equal to $\overline{\Lambda}_k$, i.e. $\frac{dP}{d\overline{P}}\Big|_{\Xi_k} = \overline{\Lambda}_k$. Obviously, with this specification, in order for the construction of P using a product of the factors $\overline{\lambda}_l$ to be well-defined, the restriction $\langle \sigma, X_{l-1}\rangle \neq 0$ needs to hold.

As was mentioned in the discrete observations case, it is standard to make the assumption that the observation process has nonsingular noise, which ensures that the above formulation of P is well-defined. Furthermore, this assumption is appropriate not only from a technical perspective but from a modeling viewpoint as well.

To see this, note that if the components of the function $c(\cdot)$ are all different and $\langle \sigma, X_k\rangle$ is indeed equal to zero, then observing $y_{k+1} = c_r$, for example, implies with certainty that $X_k = e_r$. That is, one only needs to wait for one time period before learning the state of the signal process in the immediate past with certainty, which contradicts the concept of a hidden Markov model (i.e. the underlying signal process is unobservable).

As we did in the preceding sections, we now prove several properties of the probability measure P in order to make sure that its specification above correctly changes the distribution of the observation process. Unlike results in the previous sections, Elliott et al. (1995) do not provide any details on the results that will follow in this section. In the statement and proofs of the theorems below, we have borrowed ideas from the example of a measure change for linear systems presented in Aggoun & Elliott (2004) with some adjustments.

First, we combine into one theorem some important properties for $\overline{\Lambda}_k$ and $\overline{\lambda}_k$ under $\overline{P}$:

Lemma 3.4.13 (Properties of $\overline{\Lambda}_k$ and $\overline{\lambda}_k$ under $\overline{P}$) *The process* $\{\overline{\Lambda}_k\}, k \in \mathbb{N}$, *is a* $\overline{P}$*-martingale with respect to the filtration* $\{\Xi_k\}$.

See the Appendix for a proof.

Then, we make sure P changes the distribution of the observation process appropriately:

Lemma 3.4.14 (Distribution of w_k under P) *Under the probability measure P, the $\{w_k\}$, $k \in \mathbb{N}$, is a sequence of $N(0,1)$ i.i.d. random variables.*

The proof follows in the Appendix.

In the remainder of the sections on Bayesian estimation, for greater simplicity, we shall work under $\overline{P}$ and shall then translate the estimation results back to P. As was discussed above, the main tenet of the reference probability method is to suppose that all processes are initially defined on an "ideal" probability space $(\Omega, \Sigma, \overline{P})$. In the section dealing with the change of measure, it was shown that such a probability measure exists, and its construction and properties were discussed at length.

Then, in the section on the reverse change of measure, we discussed how to construct the probability measure P, starting from $\overline{P}$, such that under P the observed real-world model dynamics, as summarized in equation (3.60), will hold. This allows the estimation problem to be solved under the idealized probability measure $\overline{P}$, taking full advantage of the independence of the observation process under $\overline{P}$, and then the results are translated back to the real-world measure P by means of the conditional Bayes theorem (3.4.3) or a version of it.

In the following two sections, we follow the structure of our discussion of the discrete observations case: firstly, we give formulae for the calculation of unnormalized state estimates, and then we normalize them in order to obtain the conditional probabilities of the signal process being in a particular state, given a realization of the observation process. We also extend our analysis to include a general formula for the unnormalized recursive filter for a sequence of arbitrary adapted random variables.

3.4.2.11 Unnormalized Estimates and Conditional Probabilities: Continuous-Observations Case

In order to estimate the state of the signal process, given a realization of the observation process and an assumed model of the dynamics of the state and observation processes, we wish to evaluate the expectation $E[X_k \,|\Upsilon_k]$. Since the signal process X_k is a simple random variable, we shall use the terms "expectation" and "probability" interchangeably. This, however, will not be true in the subsequent section.

From a version of the conditional Bayes theorem, we have:

$$E[X_k \,|\Upsilon_k] = \frac{\overline{E}\left[\Lambda_k X_k \,|\Upsilon_k\right]}{\overline{E}\left[\Lambda_k \,|\Upsilon_k\right]}. \tag{3.67}$$

Denote the vector of unnormalized conditional probabilities of the signal process being in each of its possible states at time k, given the history of observations up until and including time k, by:

$$q_k := \overline{E}\left[\Lambda_k X_k \left| \Upsilon_k \right.\right] \in \mathbb{R}^n. \tag{3.68}$$

Also, define the following quantities:

$$\begin{aligned}
\psi_j\left(y_{k+1}\right) &:= \frac{\phi\left(\frac{y_{k+1}-\langle c,e_j\rangle}{\langle\sigma,e_j\rangle}\right)}{\langle\sigma,e_j\rangle\phi\left(y_{k+1}\right)}, && (3.69)\\
B\left(y_{k+1}\right) &:= \begin{bmatrix} \psi_1\left(y_{k+1}\right) & & \\ & \ddots & \\ & & \psi_N\left(y_{k+1}\right) \end{bmatrix}, && (3.70)\\
\Gamma_{0,k} &:= B\left(y_k\right) AB\left(y_{k-1}\right) AB\left(y_{k-2}\right) \ldots AB\left(y_1\right). && (3.71)
\end{aligned}$$

Note that the definition of $\psi_j\left(y_{k+1}\right)$ above is very similar to the definition of $\overline{\lambda}_{k+1}$ given above. The only difference is that instead of relying on the state of the signal process X_k to choose a component of $c\left(\cdot\right)$ and $\sigma\left(\cdot\right)$ respectively, each $\psi_j\left(y_{k+1}\right)$ explicitly chooses the j-th element of these vectors. These $\psi_j\left(y_{k+1}\right)$ are then arranged along the main diagonal of the $N \times N$ diagonal matrix $B\left(y_{k+1}\right)$. Finally, $\Gamma_{0,k}$ defines the recursive relation that can be used in order to obtain the conditional probability of the signal process at time k starting from the prior q_0.

Using this notation, the following result can be derived:

Lemma 3.4.15 (Recursive Filter - Continuous-Range Observations) *The vector of unnormalized conditional state probabilities at time $k+1$, q_{k+1}, can be computed from the conditional state probabilities at time k, q_k, using the recursion:*

$$\begin{aligned}
q_{k+1} &= B\left(y_{k+1}\right) Aq_k && (3.72)\\
&= \Gamma_{0,k}q_0.
\end{aligned}$$

The proof is provided in the Appendix.

It is straightforward to find an expression for the conditional probability that the signal process is in a specific state simply by computing the result from the matrix and vector multiplications in equation (3.72). Denote by $q_k\left(e_j\right) := \overline{E}\left[\Lambda_k\langle X_k, e_j\rangle \left| \Upsilon_k \right.\right]$ the conditional probability of the signal process being in state j at time k. Then, $q_{k+1}\left(e_j\right)$ is the jth element of vector q_{k+1}. A short calculation shows that the conditional probability of the signal process being in state i at time $k+1$ is given by:

$$q_{k+1}\left(e_i\right) = \sum_{j=1}^{N} \psi_i\left(y_{k+1}\right) a_{ij} q_k\left(e_j\right). \tag{3.73}$$

Intuitively, this result is equivalent to the formula we obtained for the model with discrete-range observations. To see this, consider the interpretation of each of the terms in equation (3.73). Under the probability measure P, the noise terms $\{w_k\}$ are a sequence of $N(0,1)$ i.i.d. random variables. Since we denoted the standard normal density by $\phi(\cdot)$, the term $\phi(w_k)$ gives the probability of occurrence of the noise term w_k. Conversely, under $\overline{P}$, the observation process $\{y_k\}$ is a sequence of $N(0,1)$ i.i.d. random variables, and $\phi(y_k)$ gives the probability of occurrence of the observation process value under $\overline{P}$.

Consequently, what the quantity $\psi_j(y_{k+1}) = \frac{\phi\left(\frac{y_{k+1}-\langle c,e_j\rangle}{\langle \sigma,e_j\rangle}\right)}{\langle \sigma,e_j\rangle \phi(y_{k+1})}$ really represents is the probability of occurrence of the value of the observation process y_k at time k under the real-world probability measure P. The other two terms in equation (3.73) have the same interpretation as before. That is, equation (3.73) computes the probability of the signal process being in state i at time $k+1$ by taking the conditional probability of the signal process being in each one of its states j at time k, multiplying this by the transition probability from each of the states j to state i, further multiplying this product by the probability of observing the realized value of the observation process at time $k+1$, and finally summing these products over each of the N possible states of the signal process. The similarities between the unnormalized state estimates for the two cases – discrete-range and continuous-range observations – is summarized in the tables below.

Table 3.3: Unnormalized State Estimates: Discrete-Range Observations

Vector Form	$q_{k+1} = A\; diag\,(q_k)\, c_{(\cdot)}\,(Y_{k+1})$
Scalar Form	$q_{k+1}(e_r) = \sum_{j=1}^{N} q_k(e_j)\, a_{rj} M \prod_{i=1}^{M} c_{ij}^{Y_{k+1}^i}$

Table 3.4: Unnormalized State Estimates: Continuous-Range Observations

Vector Form	$q_{k+1} = B(y_{k+1})\, A q_k$
Scalar Form	$q_{k+1}(e_i) = \sum_{j=1}^{N} \psi_i(y_{k+1})\, a_{ij} q_k(e_j)$

Once the unnormalized state estimates are computed, then it is trivial to obtain the conditional probabilities of the signal process being in any of its possible states. This is achieved by merely normalizing the unnormalized estimates by dividing them by $\overline{E}\left[\overline{\Lambda}_k \,|\Upsilon_k\right]$. This is indeed a normalization since, as in the discrete observations case, we have:

$$\sum_{i=1}^{N} q_k(e_i) = \overline{E}\left[\overline{\Lambda}_k \sum_{i=1}^{N} \langle X_k, e_i\rangle \,|\Upsilon_k\right] = \langle q_k, \underline{1}\rangle = \overline{E}\left[\overline{\Lambda}_k \,|\Upsilon_k\right],$$

since $\sum_{i=1}^{N}\langle X_k, e_i\rangle = 1$. In the above, as in the previous sections, the notation $\underline{1}$ denotes a vector, whose elements are all one: $(1, 1, \dots, 1)' \in \mathbb{R}^N$. As before, we now present a generalized version of the unnormalized state estimate formula and show that equation (3.72) is a special case of this more general result.

3.4.2.12 A General Unnormalized Recursive Filter: Continuous-Observations Case

In this section we extend the analysis above and consider the conditional expectations of arbitrary random variables, specified in a much more flexible way. We will keep most of the notation from the equivalent section, which considered this problem within the context of discrete-range observations.

Let $\{H_k\}, k \in \mathbb{N}$, be any sequence of adapted, for example to the filtration $\{\Xi_k\}$, random variables. We shall write:

$$\gamma_k (H_k) = \overline{E}\left[\Lambda_k H_k \,|\Upsilon_k\right]. \tag{3.74}$$

That is, $\gamma_k (H_k)$ is the unnormalized conditional expectation of the random variable H_k given Υ_k. Note, that unlike the previous section, here we cannot consider $\gamma_k (H_k)$ to be a conditional probability.

Using the conditional Bayes theorem for sequences of random variables in lemma (1.4.4), we can write:

$$\hat{H}_k := E\left[H_k \,|\Upsilon_k\right] = \frac{\overline{E}\left[\Lambda_k H_k \,|\Upsilon_k\right]}{\overline{E}\left[\Lambda_k \,|\Upsilon_k\right]} = \frac{\gamma_k (H_k)}{\gamma_k (1)}. \tag{3.75}$$

Again, we shall assume that the initial distribution for H is given: $\gamma_0 (H_0) = E[H_0]$. This will provide the initial value for the recursions.

In order to simplify calculations, we shall assume that $\{H_k\}, k \in \mathbb{N}$, is a sequence of scalar random variables. Therefore, we can write the first differences of the random variables as:

$$\Delta H_{k+1} = H_{k+1} - H_k, \quad H_{k+1} = H_k + \Delta H_{k+1},$$

and therefore:

$$\gamma_{k+1} (H_{k+1}) = \overline{E}\left[\Lambda_{k+1} H_k \,|\Upsilon_k\right] + \overline{E}\left[\Lambda_{k+1}\Delta H_{k+1} \,|\Upsilon_{k+1}\right].$$

Consider the first term on the right-hand side:

$$\begin{aligned}\overline{E}\left[\Lambda_{k+1} H_k \,|\Upsilon_k\right] &= \overline{E}\left[\Lambda_k H_k \overline{\lambda}_{k+1} \,|\Upsilon_{k+1}\right] \\ &= \overline{E}\left[\Lambda_k H_k \frac{\phi\left(\frac{y_{k+1} - \langle c, X_k\rangle}{\langle \sigma, X_k\rangle}\right)}{\langle \sigma, X_k\rangle \phi\left(y_{k+1}\right)} \,|\Upsilon_{k+1}\right].\end{aligned} \tag{3.76}$$

In order to avoid cumbersome notation, write:

$$\Gamma^{(\cdot)}\left(y_k\right) := \frac{\phi\left(\frac{y_k - c_{(\cdot)}}{\sigma_{(\cdot)}}\right) e_{(\cdot)}}{\sigma_{(\cdot)}\phi\left(y_k\right)}.$$

We have already encountered the fact that $\sum_{i=1}^{N}\langle X_k, e_i\rangle = 1$ on multiple occasions. Also, note that the y_n, $1 \leq n \leq k+1$, are known, since we are conditioning on Υ_{k+1}, and this includes all observations of the process $\{y_k\}$ up until and including time $k+1$. Consequently:

$$\begin{aligned} \overline{E}\left[\overline{\Lambda}_{k+1} H_k \,|\Upsilon_{k+1}\right] &= \sum_{i=1}^{N} \overline{E}\left[\overline{\Lambda}_k H_k \langle X_k, \Gamma^i\left(y_{k+1}\right)\rangle \,|\Upsilon_{k+1}\right] \\ &= \sum_{i=1}^{N} \langle \gamma_k\left(H_k X_k\right), \Gamma^i\left(y_{k+1}\right)\rangle. \end{aligned}$$

The first equality is just an equivalent representation of the quantity in equation (3.76): $\Gamma^i\left(y_{k+1}\right)$ is a vector of dimension N, whose elements are all zero apart from the ith element, which is equal to $\frac{\phi\left(\frac{y_{k+1} - \langle c, X_k\rangle}{\langle\sigma, X_k\rangle}\right)}{\langle\sigma, X_k\rangle\phi(y_{k+1})}$. On the other hand, X_k has the same structure: all of its elements apart from one are zero. The only nonzero element is equal to one. So, the inner product will evaluate to zero in all states but the one that X_k is in. The second equality simply uses the fact that the only source of uncertainty in the first equality is X_k and applies the definition of $\gamma_k\left(\cdot\right)$ inside the inner product operator.

Obviously, this approach to the estimation of the unnormalized conditional expectation of H_{k+1} brings about the same problem that was established in the discrete observations case. That is, the estimate for the quantity $\gamma_{k+1}\left(H_{k+1}\right)$ involves the term $\gamma_k\left(H_k X_k\right)$, rather than $\gamma_k\left(H_k\right)$, which means that it is not possible to formulate a recursive relationship for the estimation of $\gamma_{k+1}\left(H_{k+1}\right)$ using this approach.

An alternative approach, which circumvents this technical difficulty, is to examine the recursion for the quantity $\gamma_{k+1}\left(H_{k+1}X_{k+1}\right)$ instead. By an argument similar to the above, it can be shown that the estimate for the unnormalized conditional expectation $\gamma_{k+1}\left(H_{k+1}X_{k+1}\right)$ introduces the term:

$$\gamma_k\left(H_k X_k X_k^{'}\right) = \sum_{i=1}^{N} \langle \gamma_k\left(H_k X_k\right), e_i\rangle e_i e_i^{'}.$$

That is, the estimate for $\gamma_{k+1}\left(H_{k+1}X_{k+1}\right)$ can be formulated as a recursive relation and expressed in terms of $\gamma_k\left(H_k X_k\right)$, together with other terms.

However, since the initial objective was to find a recursive formulation for the unnormalized conditional expectation $\gamma_{k+1}\left(H_{k+1}\right)$, and not for $\gamma_{k+1}\left(H_{k+1}X_{k+1}\right)$, a procedure for the elimination of the additional X_{k+1} must be introduced. It is not difficult to accomplish this by taking advantage of the structure of X_{k+1}. We proceed in the same way as in the discrete observations case. Denote by $\underline{1}$ the vector with each of its elements being equal to one: $\underline{1} := (1, 1, \ldots, 1)' \in \mathbb{R}^N$. This allows us to represent the sum of the elements of X_{k+1} by $\sum_{i=1}^{N} \langle X_{k+1}, e_i \rangle = \langle X_{k+1}, \underline{1} \rangle = 1$. Consequently we can write:

$$
\begin{aligned}
\langle \gamma_{k+1} \left(H_{k+1} X_{k+1} \right), \underline{1} \rangle &= \gamma_{k+1} \left(\langle H_{k+1} X_{k+1}, \underline{1} \rangle \right) \\
&= \gamma_{k+1} \left(H_{k+1} \langle X_{k+1}, \underline{1} \rangle \right) = \gamma_{k+1} \left(H_{k+1} \right). \qquad (3.77)
\end{aligned}
$$

In other words, once the unnormalized conditional expectation $\gamma_{k+1} \left(H_{k+1} X_{k+1} \right)$ is computed, the estimate $\gamma_{k+1} \left(H_{k+1} \right)$ is obtained by summing the elements of $\gamma_{k+1} \left(H_{k+1} X_{k+1} \right)$.

Next, consider the denominator in equation (3.75). The methodology shown in equation (3.77) can be used in order to evaluate the quantity $\overline{E} \left[\Lambda_k \,|\Upsilon_k \right]$ as well. To this end, by taking $H_k = 1$ in equation (3.77), we obtain:

$$
\begin{aligned}
\gamma_k (1) &= \gamma_k \left(\langle X_k, \underline{1} \rangle \right) = \langle \gamma_k \left(X_k \right), \underline{1} \rangle \\
&= \overline{E} \left[\Lambda_k \,|\Upsilon_k \right].
\end{aligned}
$$

That is, even in the case when we are considering the conditional expectation of an arbitrary random variable H_k, the normalizing factor to be applied to the unnormalized conditional expectation $\gamma_k \left(H_k \right)$ is still the sum of the elements of the unnormalized conditional expectation $\gamma_k \left(X_k \right)$, which was the subject of the discussion in the previous section.

We finalize the discussion in this section by formulating a more specific, though general, specification of the process $\{H_k\}$ and deriving a recursive representation of its conditional expectation. The same terminology and notation as in previous sections will be used: i.e. a process $\{\phi_k\}$ will be called predictable with respect to the filtration Ξ_k if ϕ_k is measurable with respect to Ξ_{k-1} at each time point k. The following theorem gives a concrete specification of the process $\{H_k\}$ and derives a recursive formula for its conditional expectation.

Theorem 3.4.16 (General Recursive Filter - Continuous-Range Observations) *Let $\{H_k\}$ be a scalar Ξ-adapted process of the form:*

$$
H_{k+1} = H_k + \alpha_{k+1} + \langle \beta_{k+1}, V_{k+1} \rangle + \delta_{k+1} f \left(y_{k+1} \right), \quad k \geq 1,
$$

where H_0 is Σ_0-measurable. Here, $V_{k+1} = X_{k+1} - AX_k$, $f(\cdot)$ is an arbitrary scalar-valued function, and α, β, δ are Ξ-predictable processes of appropriate dimensions; i.e. α and δ are scalar-valued, while β is an N-dimensional vector *process. Then, we have the following recursive relation:*

$$
\begin{aligned}
\gamma_{k+1}\left(H_{k+1}X_{k+1}\right) &:= \gamma_{k+1,k+1}\left(H_{k+1}\right) \\
&= \sum_{i=1}^{N}\Big\{\langle\gamma_k\left(H_kX_k\right),\Gamma^i\left(y_{k+1}\right)\rangle a_i \\
&+ \gamma_k\left(\alpha_{k+1}\langle X_k,\Gamma^i\left(y_{k+1}\right)\rangle\right)a_i \\
&+ \gamma_k\left(\delta_{k+1}\langle X_k,\Gamma^i\left(y_{k+1}\right)\rangle\right)f\left(y_{k+1}\right)a_i \\
&+ \left(diag\left(a_i\right)-a_ia_i^{'}\right)\gamma_k\left(\beta_{k+1}\langle X_k,\Gamma^i\left(y_{k+1}\right)\rangle\right)\Big\},
\end{aligned} \tag{3.78}
$$

where $a_i = Ae_i$.

As discussed above, the final step in obtaining $\gamma_{k+1}\left(H_{k+1}\right)$ from equation (3.78) is to sum all elements of $\gamma_{k+1}\left(H_{k+1}X_{k+1}\right)$. This unnormalized estimate is then divided by the sum of the components of $\gamma_{k+1}\left(X_{k+1}\right)$ in order to obtain the conditional expectation of the random variable H_{k+1}: $E\left[H_{k+1}\left|\Upsilon_{k+1}\right.\right] = \frac{\gamma_{k+1}(H_{k+1})}{\gamma_{k+1}(1)}$.

A brief examination suffices to see that equation (3.78) gives the same result as theorem (3.4.9), with the only exception that the probabilities of observing the realizations of the observation process are different. That is, $\Gamma^i\left(y_{k+1}\right)$ is substituted in equation (3.78) for $c_j\left(Y_{k+1}\right)e_j$ in equation (3.57). The proof of theorem (3.4.16) also proceeds in exactly the same way as the proof of theorem (3.4.9) and is therefore omitted. Furthermore, taking $H_k = H_0 = 1$ and $\alpha_{k+1} = 0, \beta_{k+1} = 0$, and $\delta_{k+1} = 0$ in theorem (3.4.16) yields the special case result discussed in lemma (3.4.15) in the previous section.

Having discussed the relevant theoretical background to the problem of recursive state estimation in hidden Markov models, we now proceed to the application of the result in lemma (3.4.15) to the problem of estimating conditional probabilities of the states of each industry from observations of the dividend process realizations. We pay attention to some technical difficulties in applying equation (3.72) within the context of our model. Furthermore, we also provide a short comparison of two alternative Bayesian updating algorithms and their computational characteristics.

3.4.2.13 Comparison of Two Bayesian Updating Algorithms and Calculation of Fundamental Values

Apart from the theory on state estimation presented above, the classic works on estimation and control of hidden Markov models discuss a number of additional estimation problems. For instance, Elliott et al. (1995) also demonstrate derivations for smoothed state estimates, estimators for the number of state transitions, for the amount of time spent in a particular state, as well as estimators for the observation process. The expectation maximization (EM) parameter reestimation procedure is also presented for the cases, in which the model parameters are not given a priori, but have to be estimated as well.

Our task, however, is much narrower and conceptually simpler. Firstly, note that according to the assumptions of our model, at time t_n investors can only observe dividend realizations up until and including time t_n. Since the rest of the dividend realizations until terminal time T are unknown, a derivation of smoothed state estimates is not possible. Furthermore, we assume that investors know the specification of the dividend generating processes, as well as the model parameters, and only have to estimate the probability of being in each of the states of the industry given dividend data. Therefore, EM parameter reestimation is unnecessary within the context of our model, and we devote this section solely to a discussion on the application of the theoretical results, presented in the sections above, to the estimation of regime probabilities for the industry, conditional on realizations of the dividend process. Albeit theoretically straightforward, one may encounter certain technical difficulties with this task.

As in the theoretical sections above, let us begin with the specification of the state and observation processes in our model. This was done in the section on specifying the uncertainty in the economy, so the formulations we need will be reiterated here.

The state process in our model describes the evolution of the two states of each industry over time. It conforms to the theoretical description provided in the set of equations (3.60): $X_{k+1} = AX_k + V_{k+1}$. That is, the evolution of the state of each industry from time k to $k+1$ can be described in terms of a probability distribution plus an unbiased error term. Since we modeled the time between regime switches as an exponentially distributed random variable, the probability distribution, specified by the matrix of transition probabilities A, has the following form:

$$A = \begin{bmatrix} e^{-\overline{\lambda}(t_{n+1}-t_n)} & 1 - e^{-\overline{\lambda}(t_{n+1}-t_n)} \\ 1 - e^{-\underline{\lambda}(t_{n+1}-t_n)} & e^{-\underline{\lambda}(t_{n+1}-t_n)} \end{bmatrix},$$

where:

$$\begin{aligned} P\left(\mu_{t_{n+1}} = \overline{\mu} \,|\mu_{t_n} = \overline{\mu}\right) &= e^{-\overline{\lambda}(t_{n+1}-t_n)} \\ P\left(\mu_{t_{n+1}} = \underline{\mu} \,|\mu_{t_n} = \underline{\mu}\right) &= e^{-\underline{\lambda}(t_{n+1}-t_n)} \\ P\left(\mu_{t_{n+1}} = \overline{\mu} \,|\mu_{t_n} = \underline{\mu}\right) &= 1 - e^{-\underline{\lambda}(t_{n+1}-t_n)} \\ P\left(\mu_{t_{n+1}} = \underline{\mu} \,|\mu_{t_n} = \overline{\mu}\right) &= 1 - e^{-\overline{\lambda}(t_{n+1}-t_n)}, \end{aligned}$$

and where $\overline{\lambda}$ and $\underline{\lambda}$ are two different intensity parameters that model the fact that the economy generally spends differing amounts of time in its two regimes.

Next, in order to specify the observation process, we need discretized versions of the stochastic differential equations that describe the dividend generating processes. These were provided in the section on specifying the uncertainty in the economy. We used a mean-reverting square root process and a geometric Ornstein-Uhlenbeck process in order to model the evolution of the dividend intensity over time, as well as Milstein's scheme in order to discretize these processes. Thus, the discretized version of the mean-reverting square root process that we obtained was:

$$\begin{aligned} \delta_{k,t_{n+1}} &= \delta_{k,t_n} + \alpha\left(\mu_{k,t_n} - \delta_{k,t_n}\right)\left(t_{n+1} - t_n\right) + \beta\sqrt{\delta_{k,t_n}}\left(W_{t_{n+1}} - W_{t_n}\right) \\ &+ \frac{1}{4}\beta^2\left(\left(W_{t_{n+1}} - W_{t_n}\right)^2 - \left(t_{n+1} - t_n\right)\right), \end{aligned} \tag{3.79}$$

and the discretized version of the geometric Ornstein-Uhlenbeck was:

$$\begin{aligned} \delta_{k,t_{n+1}} &= \delta_{k,t_n} + \alpha\left(\mu_{k,t_n} - \delta_{k,t_n}\right)\delta_{k,t_n}\left(t_{n+1} - t_n\right) + \beta\delta_{k,t_n}\left(W_{t_{n+1}} - W_{t_n}\right) \\ &+ \frac{1}{2}\beta^2\delta_{k,t_n}\left(\left(W_{t_{n+1}} - W_{t_n}\right)^2 - \left(t_{n+1} - t_n\right)\right). \end{aligned} \tag{3.80}$$

Although we considered a very general and flexible hidden Markov model in the theoretical sections above, it quickly becomes evident that the theoretical results derived there are not immediately applicable to our model. This is so, because even though we discussed a general specification of the state process in terms of the sequence of arbitrary random variables $\{H_k\}$, the theoretical sources had little to offer in terms of generality of the observation process. Thus, the formulation of the discretized dividend processes in equations (3.79) and (3.80) clearly do not conform to the assumed theoretical specification in equation (3.60). In both of these discretizations, the observation process has the form of both an autoregressive process and a hidden Markov model. Furthermore, the last terms on the right clearly do not conform to the theoretical assumption of unbiased and independent, normally distributed error terms.

A quick way to circumvent this problem is to consider the Euler-Maruyama method for discretization of stochastic differential equations (see e.g. Kloeden & Platen (1999)) instead. This will only include the first two terms on the right-hand side in equations (3.79) and (3.80), which will bring the specification of the discretized dividend process closer to the structure of the observation process assumed in the theoretical sources. Furthermore, we can focus on the dividend intensity increment $\delta_{k,t_{n+1}} - \delta_{k,t_n}$ instead of the dividend intensity in the next time period. This leads to:

$$\delta_{k,t_{n+1}} - \delta_{k,t_n} = \alpha\left(\mu_{k,t_n} - \delta_{k,t_n}\right)\left(t_{n+1} - t_n\right) + \beta\sqrt{\delta_{k,t_n}}\left(W_{t_{n+1}} - W_{t_n}\right) \tag{3.81}$$

for the mean-reverting square root process, and to:

$$\delta_{k,t_{n+1}} - \delta_{k,t_n} = \alpha\left(\mu_{k,t_n} - \delta_{k,t_n}\right)\delta_{k,t_n}\left(t_{n+1} - t_n\right) + \beta\delta_{k,t_n}\left(W_{t_{n+1}} - W_{t_n}\right) \tag{3.82}$$

for the geometric Ornstein-Uhlenbeck process.

Equations (3.81) and (3.82) now conform to the structure of the observation process defined in equation (3.60): in both equations, the first term on the right-hand side is indeed a scalar-valued function of the the state process $\{\mu_{k,t_n}\}$, while the second term is independent and identically normally distributed with mean 0. Depending on the process, the variance of the unbiased error term is either $\beta^2\delta_{k,t_n}(t_{n+1}-t_n)$ for the mean-reverting square root process, or $\beta^2\delta_{k,t_n}^2(t_{n+1}-t_n)$ for the geometric Ornstein-Uhlenbeck process. That is, the terms w_{k+1} and $\sigma(X_k)$ in equation (3.60) correspond to the terms $W_{t_{n+1}}-W_{t_n}$ and $\beta\sqrt{\delta_{k,t_n}}$ or $\beta\delta_{k,t_n}$ in equations (3.81) and (3.82) respectively. Note that in this case the diffusion term $\sigma(X_k)$ is a constant parameter rather than a function of the state process.

Having specified the matrix of transition probabilities and shown that the discretized dividend process conforms to the theoretical structure of the observation process, the last necessary ingredient for the application of equation (3.72) are the probabilities of observing a certain realization of the dividend process. Using equations (3.81) and (3.82), we can write:

$$\delta_{k,t_{n+1}}-\delta_{k,t_n}\sim N\left(\alpha\left(\mu_{k,t_n}-\delta_{k,t_n}\right)\left(t_{n+1}-t_n\right),\beta^2\delta_{k,t_n}\left(t_{n+1}-t_n\right)\right) \tag{3.83}$$

for the distribution of the dividend intensity increments when using the mean-reverting square root process, and:

$$\delta_{k,t_{n+1}}-\delta_{k,t_n}\sim N\left(\alpha\left(\mu_{k,t_n}-\delta_{k,t_n}\right)\delta_{k,t_n}\left(t_{n+1}-t_n\right),\beta^2\delta_{k,t_n}^2\left(t_{n+1}-t_n\right)\right) \tag{3.84}$$

for the geometric Ornstein-Uhlenbeck process. In other words, the increments of the dividend intensities are normally distributed with the mean being equal to the drift term and the variance equal to the square of the diffusion term in the corresponding stochastic differential equations, which describe the evolution of the dividend intensity over time.

Assume we are working with the mean-reverting square root process. Using equation (3.83), the probability of observing a dividend intensity increment at least as large as the one that is realized at time t_{n+1}, on condition that the industry is in state $\overline{\mu}$, is:

$$\frac{1}{\beta\sqrt{2\pi\delta_{k,t_n}\left(t_{n+1}-t_n\right)}}exp\left(-\frac{\left(\delta_{k,t_{n+1}}-\delta_{k,t_n}-\alpha\left(\overline{\mu}-\delta_{k,t_n}\right)\left(t_{n+1}-t_n\right)\right)^2}{2\beta^2\delta_{k,t_n}\left(t_{n+1}-t_n\right)}\right).$$

Some authors (see e.g. Fraser (2009), Rabiner (1989)) refer to such conditional probabilities as "emission probabilities".

We now proceed to the application of equation (3.72) within the context of our model. Denote by $\tilde{p}_{\overline{\mu},t_{n+1}}$ the unnormalized conditional probability of being in state $\overline{\mu}$ at time t_{n+1} given the history of dividends up until and including time t_{n+1}. Similarly, denote by $\tilde{p}_{\underline{\mu},t_{n+1}}$ the unnormalized conditional probability of being in state $\underline{\mu}$ at time t_{n+1}. Some sources refer to these quantities as "forward probabilities", stressing their role in the so called forward algorithm used in implementing hidden Markov models. Having obtained the two unnormalized probabilities above, the normalized conditional probability of being in state $\overline{\mu}$ at time t_{n+1} is given by:

$$p_{t_{n+1}} = \frac{\tilde{p}_{\overline{\mu},t_{n+1}}}{\tilde{p}_{\overline{\mu},t_{n+1}} + \tilde{p}_{\underline{\mu},t_{n+1}}}. \tag{3.85}$$

Since we only have two states in each industry, it is sufficient to give only the probability of one of the regimes.

Using the notation above, and assuming we are working with the mean-reverting square root process, we can write the recursive expressions for the unnormalized conditional probabilities $\tilde{p}$ as follows:

$$\begin{aligned}
\tilde{p}_{\overline{\mu},t_{n+1}} &= \frac{1}{\beta\sqrt{2\pi\delta_{k,t_n}\left(t_{n+1}-t_n\right)}} exp\left(-\frac{\left(\delta_{k,t_{n+1}} - \delta_{k,tn} - \alpha\left(\overline{\mu} - \delta_{k,t_n}\right)\left(t_{n+1}-t_n\right)\right)^2}{2\beta^2\delta_{k,t_n}\left(t_{n+1}-t_n\right)}\right) \\
&\times \left(p_{t_n} e^{-\overline{\lambda}(t_{n+1}-t_n)}\right) \\
&+ \frac{1}{\beta\sqrt{2\pi\delta_{k,t_n}\left(t_{n+1}-t_n\right)}} exp\left(-\frac{\left(\delta_{k,t_{n+1}} - \delta_{k,tn} - \alpha\left(\underline{\mu} - \delta_{k,t_n}\right)\left(t_{n+1}-t_n\right)\right)^2}{2\beta^2\delta_{k,t_n}\left(t_{n+1}-t_n\right)}\right) \\
&\times \left(\left(1-p_{t_n}\right)\left(1-e^{-\underline{\lambda}(t_{n+1}-t_n)}\right)\right),
\end{aligned} \tag{3.86}$$

and

$$\begin{aligned}
\tilde{p}_{\underline{\mu},t_{n+1}} &= \frac{1}{\beta\sqrt{2\pi\delta_{k,t_n}\left(t_{n+1}-t_n\right)}} exp\left(-\frac{\left(\delta_{k,t_{n+1}} - \delta_{k,tn} - \alpha\left(\overline{\mu} - \delta_{k,t_n}\right)\left(t_{n+1}-t_n\right)\right)^2}{2\beta^2\delta_{k,t_n}\left(t_{n+1}-t_n\right)}\right) \\
&\times \left(p_{t_n}\left(1-e^{-\overline{\lambda}(t_{n+1}-t_n)}\right)\right) \\
&+ \frac{1}{\beta\sqrt{2\pi\delta_{k,t_n}\left(t_{n+1}-t_n\right)}} exp\left(-\frac{\left(\delta_{k,t_{n+1}} - \delta_{k,tn} - \alpha\left(\underline{\mu} - \delta_{k,t_n}\right)\left(t_{n+1}-t_n\right)\right)^2}{2\beta^2\delta_{k,t_n}\left(t_{n+1}-t_n\right)}\right) \\
&\times \left(\left(1-p_{t_n}\right)e^{-\underline{\lambda}(t_{n+1}-t_n)}\right).
\end{aligned} \tag{3.87}$$

Equivalently, if we are working with the geometric Ornstein-Uhlenbeck process, the only change will be in the emission probabilities, in accordance with the distribution of the dividend intensity increments as shown in equation (3.84):

$$
\begin{aligned}
\tilde{p}_{\overline{\mu},t_{n+1}} &= \frac{1}{\beta\delta_{k,t_n}\sqrt{2\pi\left(t_{n+1}-t_n\right)}} exp\left(-\frac{\left(\delta_{k,t_{n+1}}-\delta_{k,tn}-\alpha\left(\overline{\mu}-\delta_{k,t_n}\right)\delta_{k,t_n}\left(t_{n+1}-t_n\right)\right)^2}{2\beta^2\delta_{k,t_n}^2\left(t_{n+1}-t_n\right)}\right)\\
&\times \left(p_{t_n}e^{-\overline{\lambda}(t_{n+1}-t_n)}\right)\\
&+ \frac{1}{\beta\delta_{k,t_n}\sqrt{2\pi\left(t_{n+1}-t_n\right)}} exp\left(-\frac{\left(\delta_{k,t_{n+1}}-\delta_{k,tn}-\alpha\left(\underline{\mu}-\delta_{k,t_n}\right)\delta_{k,t_n}\left(t_{n+1}-t_n\right)\right)^2}{2\beta^2\delta_{k,t_n}^2\left(t_{n+1}-t_n\right)}\right)\\
&\times \left(\left(1-p_{t_n}\right)\left(1-e^{-\underline{\lambda}(t_{n+1}-t_n)}\right)\right), \qquad (3.88)
\end{aligned}
$$

and respectively

$$
\begin{aligned}
\tilde{p}_{\underline{\mu},t_{n+1}} &= \frac{1}{\beta\delta_{k,t_n}\sqrt{2\pi\left(t_{n+1}-t_n\right)}} exp\left(-\frac{\left(\delta_{k,t_{n+1}}-\delta_{k,tn}-\alpha\left(\overline{\mu}-\delta_{k,t_n}\right)\delta_{k,t_n}\left(t_{n+1}-t_n\right)\right)^2}{2\beta^2\delta_{k,t_n}^2\left(t_{n+1}-t_n\right)}\right)\\
&\times \left(p_{t_n}\left(1-e^{-\overline{\lambda}(t_{n+1}-t_n)}\right)\right)\\
&+ \frac{1}{\beta\delta_{k,t_n}\sqrt{2\pi\left(t_{n+1}-t_n\right)}} exp\left(-\frac{\left(\delta_{k,t_{n+1}}-\delta_{k,tn}-\alpha\left(\underline{\mu}-\delta_{k,t_n}\right)\delta_{k,t_n}\left(t_{n+1}-t_n\right)\right)^2}{2\beta^2\delta_{k,t_n}^2\left(t_{n+1}-t_n\right)}\right)\\
&\times \left(\left(1-p_{t_n}\right)e^{-\underline{\lambda}(t_{n+1}-t_n)}\right). \qquad (3.89)
\end{aligned}
$$

These results agree with the approach taken in other theoretical sources, such as, for example, Fraser (2009) and Baum & Petrie (1966). In some sources the above recursive formulation of the unnormalized conditional probabilities is referred to as the "forward algorithm": the first stage of the forward-backward algorithm, which yields smoothed estimates for the conditional state probabilities.

It is also important to note another potential shortcoming of this approach to state estimation. Lack of generality with respect to the specification of the observation process is not the only reason that might cause the estimation approach in equations (3.86) through (3.89) to fail. Generally, the estimation methodology laid out above is valid only under the assumption that regime probabilities are Markovian. In our case this is true, due to the fact that we use the exponential distribution in order to model the probabilities of regime switches, and this distribution is memoryless. Sometimes, however, the business cycle is modeled by means of a deterministic wave-like function. This introduces additional dependence in the state process: for instance, for a certain value of the state function, one needs to know if this value comes before the apex of the business cycle or after the stage of decline has started. In such cases, the algorithm for the estimation of conditional state probabilities presented above will be inappropriate.

The above specification of the unnormalized conditional probabilities conforms well to the theoretical framework laid out in the previous sections. However, the driving force behind the exact formulation of these unnormalized conditional probabilities is the method chosen for the discretization of the dividend stochastic differential equations, since different discretizations will yield different specifications of the observation process, and will therefore affect the expressions for the emission probabilities.

Unfortunately, the specific processes we have chosen in order to model the uncertainty in the economy make it difficult to rigorously justify the above statement. Assume we are working with the mean-reverting square root process:

$$d\delta_k(t) = \alpha\left(\mu_k(t) - \delta_k(t)\right)dt + \beta\sqrt{\delta_k(t)}dW(t). \tag{3.90}$$

In general, no explicit closed form solution of the above stochastic differential equation exists (Higham & Mao (2005)). Indeed, we can try to rewrite equation (3.90) in a more general form, which would allow the application of standard analytical techniques for solving stochastic differential equations (see e.g. Øksendal (2002), chapter 12). Rewrite (3.90) as:

$$d\delta_k(t) = c\left(t, \delta_k(t)\right)\delta_k(t)\,dt + \sigma\left(t, \delta_k(t)\right)\delta_k(t)\,dW(t),$$

where

$$c\left(t, \delta_k(t)\right) = \frac{\alpha\left(\mu_k(t) - \delta_k(t)\right)}{\delta_k(t)}$$

and

$$\sigma\left(\delta_k(t)\right) = \frac{\beta}{\sqrt{\delta_k(t)}}.$$

We can then proceed with the standard argument for solving stochastic differential equations of this form:

$$\begin{aligned}
\frac{d\delta_k(t)}{\delta_k(t)} &= c\left(t, \delta_k(t)\right)dt + \sigma\left(\delta_k(t)\right)dW(t) \\
\int_0^t \frac{d\delta_k(s)}{\delta_k(s)} &= \int_0^t c\left(s, \delta_k(s)\right)ds + \int_0^t \sigma\left(\delta_k(s)\right)dW(s).
\end{aligned}$$

Let $g(t,x) \in \mathcal{C}^2\left([\,0,\infty\,) \times \mathbb{R}\right)$. That is, $g(\cdot,\cdot)$ is a twice continuously differentiable function, whose domain for the time index are the nonnegative reals.

Take $g(t,x) = ln(x)$. Then an application of Itô's lemma (Itô (1951)) yields:

$$\begin{aligned}
d\left(ln\left(\delta_k(t)\right)\right) &= \frac{d\delta_k(t)}{\delta_k(t)} - \frac{1}{2\delta_k^2(t)}\sigma^2\left(\delta_k(t)\right)\delta_k^2(t)\,dt = \frac{d\delta_k(t)}{\delta_k(t)} - \frac{1}{2}\sigma^2\left(\delta_k(t)\right)dt \\
\frac{d\delta_k(t)}{\delta_k(t)} &= d\left(ln\left(\delta_k(t)\right)\right) + \frac{1}{2}\sigma^2\left(\delta_k(t)\right)dt.
\end{aligned}$$

Integrating both sides and substituting for $\int_0^t \frac{d\delta_k(s)}{\delta_k(s)}$ yields:

$$\int_0^t c\left(s, \delta_k(s)\right)ds + \int_0^t \sigma\left(\delta_k(s)\right)dW(s) = ln\left(\frac{\delta_k(t)}{\delta_k(0)}\right) + \frac{1}{2}\int_0^t \sigma^2\left(\delta_k(s)\right)ds$$

$$\delta_k(t) = \delta_k(0)\, exp\left(\int_0^t \sigma(\delta_k(s))\, dW(s) + \int_0^t \left(c(s, \delta_k(s)) - \frac{1}{2}\sigma^2(\delta_k(s))\right) ds\right).$$

This agrees with the solution of a general Itô process in equation (12.3.32) in Øksendal (2002). We now proceed to discretize the last equation above. The stochastic integral will be taken in the Itô sense:

$$\begin{aligned} \delta_{k,t_N} &= \delta_{k,t_0} exp\left(\sum_{j=0}^{N-1} \sigma\left(\delta_{k,t_j}\right)\left(W_{t_{j+1}} - W_{t_j}\right)\right. \\ &+ \left.\sum_{j=0}^{N-1}\left[c\left(t_j, \delta_{k,t_j}\right) - \frac{1}{2}\sigma^2\left(\delta_{t_j}\right)\right](t_{j+1} - t_j)\right). \end{aligned}$$

For the stochastic term in the discretization above, the choice of a function value in each interval of the partition is significant, since, depending on whether we calculate the stochastic integral in the sense of Itô or Stratonovich, we will obtain a different value. It is a well-known fact, however, that this is not true for the Riemann integral: regardless of which function value we choose in each interval of the partition, the upper and lower Riemann sums will converge to the same value (see e.g. Higham (2001)). Hence, we might just as well write $c\left(t_{j+1}, \delta_{k,t_j}\right)$ instead of $c\left(t_j, \delta_{k,t_j}\right)$.

From the above discretization, we can concentrate on a single time step only:

$$\begin{aligned} \delta_{k,t_{n+1}} &= \delta_{k,t_n} exp\left(\sigma\left(\delta_{k,t_n}\right)\left(W_{t_{n+1}} - W_{t_n}\right)\right. \\ &+ \left.\left[c(t_{n+1}, \delta_k, t_n) - \frac{1}{2}\sigma^2(t_n, \delta_{k,t_n})\right](t_{n+1} - t_n)\right). \end{aligned}$$

Substituting for the values of the functions $c(t, \delta_k(t))$ and $\sigma(\delta_k(t))$, we obtain:

$$\begin{aligned} \delta_{k,t_{n+1}} &= \delta_{k,t_n} exp\left(\frac{\beta}{\sqrt{\delta_{k,t_n}}}\left(W_{t_{n+1}} - W_{t_n}\right)\right. \\ &+ \left.\left[\frac{\alpha\left(\mu_{k,t_{n+1}} - \delta_{k,t_n}\right)}{\delta_{k,t_n}} - \frac{1}{2}\frac{\beta^2}{\delta_{k,t_n}}\right](t_{n+1} - t_n)\right). \end{aligned}$$

Since this is not an explicit solution, the dependence on δ_{k,t_n} makes it difficult to establish the probability density of the dividend intensity increment. It is obvious, however, that the increment is certainly not normally distributed. Standard texts on mathematical finance provide a closed-form expression for the conditional density of the mean-reverting square root process. For instance, Kwok (2008) gives the probability density for the value of the process (3.90) at terminal time T, conditional on its value at the current time t as:

$$\hat{p}(\delta_k(T); \delta_k(t)) = ce^{-u-v}\left(\frac{v}{u}\right)^{\frac{q}{2}} I_q\left(2(uv)^{\frac{1}{2}}\right),$$

where

$$c=\frac{2\alpha}{\beta^{2}\left[1-e^{-\alpha(T-t)}\right]},\quad u=c\delta_{k}\left(t\right)e^{-\alpha(T-t)},\quad v=c\delta_{k}\left(t\right),\quad q=\frac{2\alpha\mu_{k}\left(t\right)}{\beta^{2}}-1,$$

and I_q denotes the modified Bessel function of the first kind of order q. Obviously, this would drastically affect the emission probabilities of $\delta_k(t)$ and would therefore significantly change the recursive Bayesian updating formulae (3.86) through (3.89).

Even though such a rigorous mathematical approach cannot be used in order to justify a different discretization of equation (3.90), we can use some of the intuition gained in the process and present a computational argument. In equation (3.81), we used the Euler-Maruyama scheme for numerical simulation of stochastic differential equations in order to solve (3.90). This solution was of the form:

$$\begin{aligned}\delta_{k,t_{n+1}}-\delta_{k,t_n} &= \alpha\int_{t_n}^{t_{n+1}}\left(\mu_k\left(s\right)-\delta_k\left(s\right)\right)ds+\int_{t_n}^{t_{n+1}}\beta\sqrt{\delta_k\left(s\right)}dW\left(s\right)\\ &\approx \alpha\left(\mu_{k,t_n}-\delta_{k,t_n}\right)\left(t_{n+1}-t_n\right)+\beta\sqrt{\delta_{k,t_n}}\left(W_{t_{n+1}}-W_{t_n}\right).\end{aligned}$$

We can use our discussion on the properties of the Riemann integral above to rewrite the above equation as:

$$\delta_{k,t_{n+1}}-\delta_{k,t_n}=\alpha\left(\mu_{k,t_{n+1}}-\delta_{k,t_n}\right)\left(t_{n+1}-t_n\right)+\beta\sqrt{\delta_{k,t_n}}\left(W_{t_{n+1}}-W_{t_n}\right). \quad (3.91)$$

Another way to justify this discretization is to approximate the process (3.90) on the interval $t\in[t_n,t_{n+1}]$ by:

$$d\hat{\delta}_k\left(t\right)=\alpha\left(\mu_{k,t_{n+1}}-\hat{\delta}_k\left(t\right)\right)dt+\beta\sqrt{\hat{\delta}_k\left(t\right)}dW\left(t\right), \quad (3.92)$$

where instead of treating the regime switching process as a function of time, we make it a constant and set it equal to the value of the function at the right interval boundary. For a small enough length of the time interval $[t_n,t_{n+1}]$, the approximation (3.92) will converge to the dividend intensity stochastic differential equation (3.90). This is so, because we model the amount of time between regime switches as an exponentially distributed random variable. Therefore, as $dt\to 0$, the probability of a regime switch occurring during this time step also tends to zero. In the discussion above, "small enough" means a time interval of a size sufficiently small to allow the solution obtained through an Euler discretization to converge to the solution of (3.90). We then discretize equation (3.92) using Euler-Maruyama's scheme and obtain the same result as in equation (3.91).

If the discretization in equation (3.91) is applied, it will change the expression for the distribution of the dividend intensity increments, and hence the formulae for the emission probabilities. Therefore, the recursive Bayesian updating formulae (3.86) through (3.89) will also change. If the discretization in equation (3.91) is used, the recursive expressions for the unnormalized conditional probabilities $\tilde{p}$ become:

$$\begin{aligned}\tilde{p}_{\overline{\mu},t_{n+1}} &= \frac{1}{\beta\sqrt{2\pi\delta_{k,t_n}\left(t_{n+1}-t_n\right)}} exp\left(-\frac{\left(\delta_{k,t_{n+1}}-\delta_{k,tn}-\alpha\left(\overline{\mu}-\delta_{k,t_n}\right)\left(t_{n+1}-t_n\right)\right)^2}{2\beta^2\delta_{k,t_n}\left(t_{n+1}-t_n\right)}\right) \\ &\times \left(p_{t_n}e^{-\overline{\lambda}(t_{n+1}-t_n)}+\left(1-p_{t_n}\right)\left(1-e^{-\underline{\lambda}(t_{n+1}-t_n)}\right)\right), \qquad (3.93)\end{aligned}$$

and

$$\begin{aligned}\tilde{p}_{\underline{\mu},t_{n+1}} &= \frac{1}{\beta\sqrt{2\pi\delta_{k,t_n}\left(t_{n+1}-t_n\right)}} exp\left(-\frac{\left(\delta_{k,t_{n+1}}-\delta_{k,tn}-\alpha\left(\underline{\mu}-\delta_{k,t_n}\right)\left(t_{n+1}-t_n\right)\right)^2}{2\beta^2\delta_{k,t_n}\left(t_{n+1}-t_n\right)}\right) \\ &\times \left(p_{t_n}\left(1-e^{-\overline{\lambda}(t_{n+1}-t_n)}\right)+\left(1-p_{t_n}\right)e^{-\underline{\lambda}(t_{n+1}-t_n)}\right). \qquad (3.94)\end{aligned}$$

Similarly, if we are working with the geometric Ornstein-Uhlenbeck process, the only difference in the expressions for the unnormalized conditional probabilities will be the different emission probabilities:

$$\begin{aligned}\tilde{p}_{\overline{\mu},t_{n+1}} &= \frac{1}{\beta\sqrt{2\pi\delta_{k,t_n}\left(t_{n+1}-t_n\right)}} exp\left(-\frac{\left(\delta_{k,t_{n+1}}-\delta_{k,tn}-\alpha\left(\overline{\mu}-\delta_{k,t_n}\right)\delta_{k,t_n}\left(t_{n+1}-t_n\right)\right)^2}{2\beta^2\delta_{k,t_n}^2\left(t_{n+1}-t_n\right)}\right) \\ &\times \left(p_{t_n}e^{-\overline{\lambda}(t_{n+1}-t_n)}+\left(1-p_{t_n}\right)\left(1-e^{-\underline{\lambda}(t_{n+1}-t_n)}\right)\right), \qquad (3.95)\end{aligned}$$

and respectively

$$\begin{aligned}\tilde{p}_{\underline{\mu},t_{n+1}} &= \frac{1}{\beta\sqrt{2\pi\delta_{k,t_n}\left(t_{n+1}-t_n\right)}} exp\left(-\frac{\left(\delta_{k,t_{n+1}}-\delta_{k,tn}-\alpha\left(\underline{\mu}-\delta_{k,t_n}\right)\delta_{k,t_n}\left(t_{n+1}-t_n\right)\right)^2}{2\beta^2\delta_{k,t_n}^2\left(t_{n+1}-t_n\right)}\right) \\ &\times \left(p_{t_n}\left(1-e^{-\overline{\lambda}(t_{n+1}-t_n)}\right)+\left(1-p_{t_n}\right)e^{-\underline{\lambda}(t_{n+1}-t_n)}\right). \qquad (3.96)\end{aligned}$$

This representation of the unnormalized conditional probabilities differs somewhat in its interpretation. In equations (3.86) through (3.89), the discretization used μ_{k,t_n}, and consequently, starting from time t_n, we had to consider the emission probabilities for each of the possible two states at time t_{n+1}. With a discretization that uses $\mu_{k,t_{n+1}}$ instead, we only need to consider the emission probability for the regime, for which we are trying to estimate a conditional probability. The term in brackets then specifies the two different states at time t_n, from which the state process could have evolved to the corresponding regime at time t_{n+1}.

The motivation for using the above discretization is that equations (3.93) through (3.96) look much more compact than their counterparts (3.86) through (3.89). Furthermore, they seem more computationally efficient, since the exponential function needs to be called only once. In order to check this initial intuition, two sets of comparative simulations of the alternative recursive Bayesian updating specifications were performed with the same set of parameters. These are given in the table below.

Table 3.5: Comparison of Bayesian Updating Algorithms: Parameters

Time Step Length	0.01
Investment Horizon in Years	10
Dividend Generating Process	Geometric Ornstein-Uhlenbeck
Number of Assets	1
$\underline{\mu}$	0.2
$\overline{\mu}$	1.25
Starting Dividend Value	1.25
Discount Rate	0.1

The two simulations were performed with 100 and 1000 sample runs respectively. Apart from the two different recursive formulae for the unnormalized conditional probabilities, all other variables were kept identical in both cases. That is, a specific scenario for the evolution of the regimes was given in both cases. Furthermore, in both sets of simulations, once generated, the Brownian increments were saved and used for both estimation approaches. This procedure ensures that the results are not contaminated by the effects of randomness and allows to compare the output from the two estimation procedures, given identical input. The simulations with 100 and 1000 sample runs were performed with different seeds for the random number generator, so as to rule out biases in the results, that may have been introduced by a realization of a particular sequence of Brownian increments.

Figure (3.1) presents the means of the two Bayesian estimators at each time point over 100 simulation runs, as well as the actual regimes. A detail worth noting, is that in order to increase the accuracy of the Bayesian estimators in the initial stages of the sample runs, we extend the simulation time horizon by 200 time steps in the very beginning. This allows the Bayesian estimators to run for a sufficient amount of time, so as to effectively "learn" the current state and eliminate any biases introduced by a potential situation, in which the priors are too far away from the actual regime. The motivation for this procedure will become evident in the section on the implementation of the model.

We refer to the set of equations (3.86) through (3.89) as "theoretical Bayesian estimators", while the set of equations (3.93) through (3.96) will be referred to as "approximate Bayesian estimators".

The graphs of the Bayesian estimators show only the conditional probability of being in the $\overline{\mu}$ regime. The difference between the two means is barely noticeable at this scale. Generally, however, the approximate Bayesian estimators

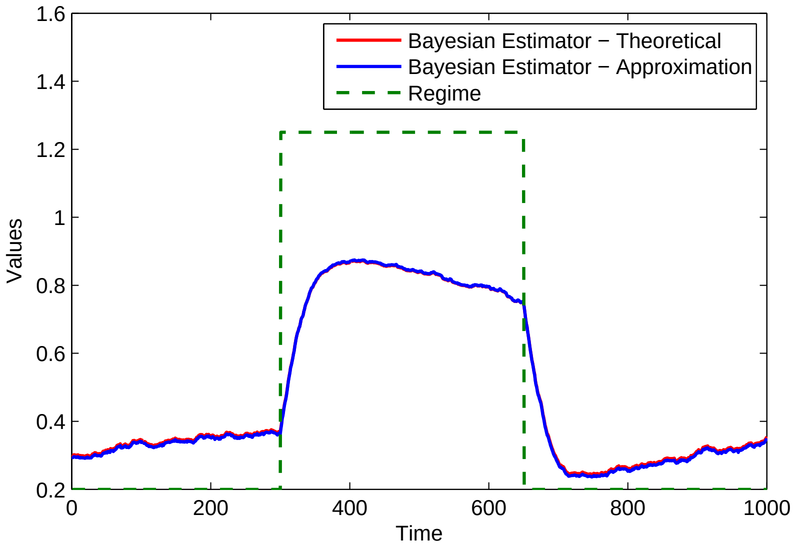


Figure 3.1: Comparison of Bayesian Estimators: 100 Simulation Runs

tend to underestimate the conditional probability of being in state $\overline{\mu}$ when the state process is in $\underline{\mu}$, and overestimate this conditional probability when the state process is in $\overline{\mu}$.

Figure (3.1) appears to support our statement that equations (3.93) through (3.96) are a reasonable approximation for the recursive Bayesian estimators (3.86) through (3.89). In order to further quantify this statement, we implement a t-test for differences in means of two samples. Figure (3.2) plots the standard error for the combined observations as a function of time. The standard error is calculated in the standard way:

$$\hat{\sigma}(t) = \sqrt{\frac{\sigma_t^2(t)}{N_t} + \frac{\sigma_a^2(t)}{N_a}},$$

where $\sigma_t^2(t)$ and $\sigma_a^2(t)$ denote the variances of the samples generated by the theoretical and approximate Bayesian estimators respectively, while N_t and N_a denote the number of observations in the corresponding samples.

The t-statistic is then given by:

$$\hat{t}(t) = \frac{\mu_t(t) - \mu_a(t)}{\hat{\sigma}(t)},$$

where $\mu_t(t)$ and $\mu_a(t)$ denote the means of the two samples. Figure (3.3) gives the absolute value of the t-statistic as a function of time. The combined group of observations has 198 degrees of freedom, and therefore the critical t-value for a two-tailed test at the 5% significance level is 1.96. Since the maximum absolute value of the t-statistic is around 0.9, the differences in the means of the two samples are not statistically significant at any point in time.

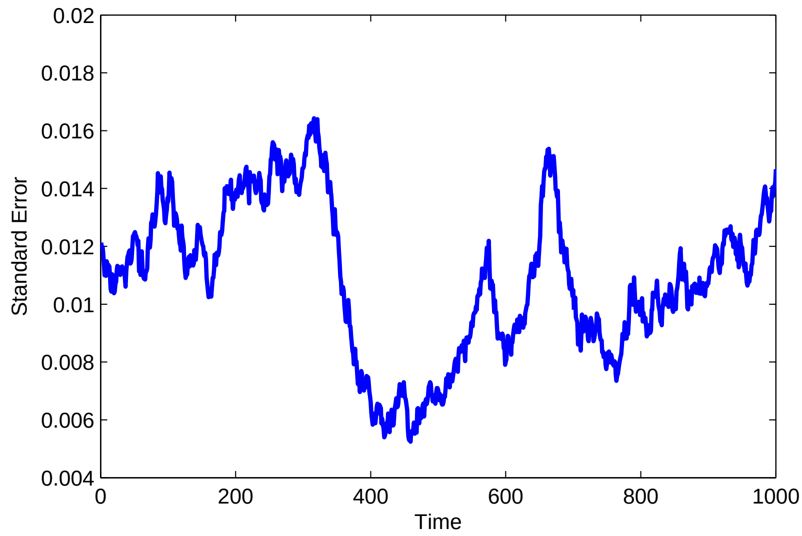


Figure 3.2: Standard Error: 100 Simulation Runs

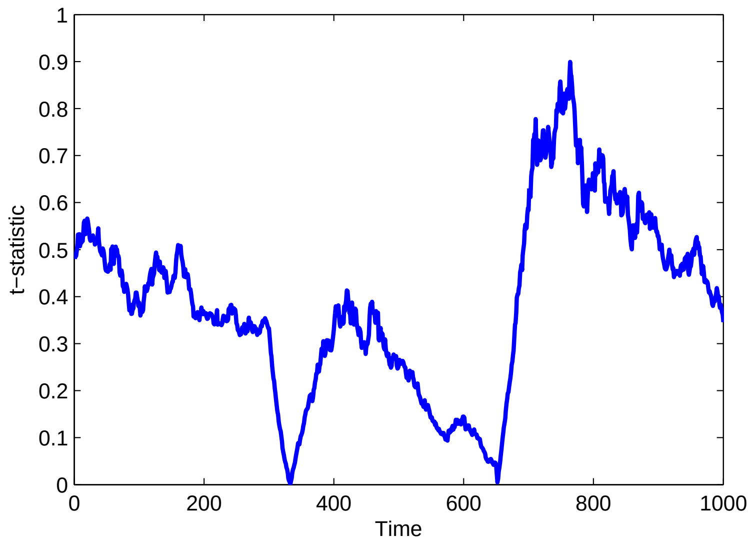


Figure 3.3: t-statistic: 100 Simulation Runs

Similarly, the next three figures present the same quantities for the simulations with 1000 sample runs. Compared to figure (3.1), in figure (3.4), averaging over more sample paths yields significantly smoother means of the conditional probabilities of being in state $\overline{\mu}$. The difference between the two means remains very small. Due to the large number of sample runs, however, in this case the standard errors are significantly lower. This leads to noticeably higher t-statistic values. Note that now there is a part of the t-statistic function, which exceeds the 1.96 critical value, and in this portion of the graph the differences between the two means are statistically significant. That is to say, if a researcher

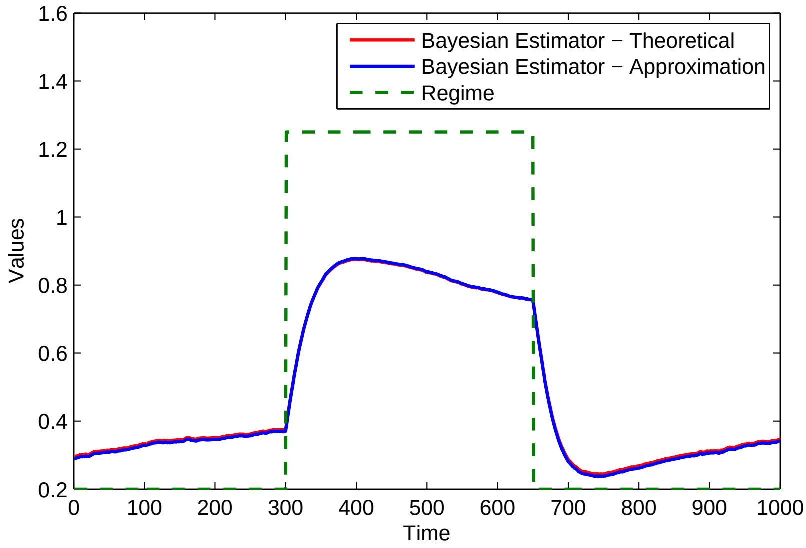


Figure 3.4: Comparison of Bayesian Estimators: 1000 Simulation Runs

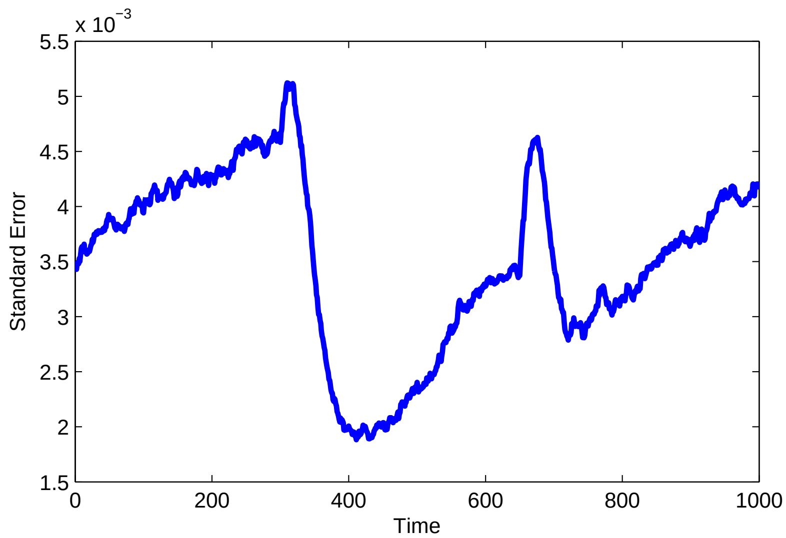


Figure 3.5: Standard Error: 1000 Simulation Runs

were to examine the data generated by the two Bayesian updating algorithms without knowing what produced the samples, it would take a sample size as large as 1000 observations in both groups before they would start perceiving the differences as statistically significant. It would take a drastic increase in sample size in order to obtain statistically significant differences at every point in time. In other words, we know that there are systematic differences between the two Bayesian estimators, but even using a time interval length of 0.01, these differences are so small that it would take very large sample sizes in order to quantify them as significant. The results from this comparison provide objective support in favour of the statement that equations (3.93) through (3.96) do

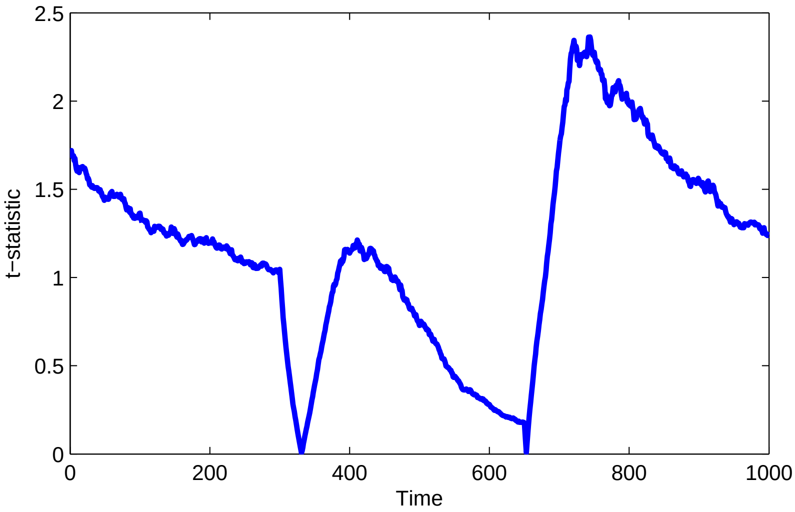


Figure 3.6: t-statistic: 1000 Simulation Runs

indeed converge to the recursive Bayesian estimators (3.86) through (3.89) as the length of the time intervals in the discretization decreases.

Recall that the main motivation behind the introduction of the approximate Bayesian estimators was to increase computational efficiency. Therefore, apart from numerically verifying the argument that the two sets of Bayesian estimators are equivalent, the other important objective of the above comparison is to examine if the approximate Bayesian estimators do indeed contribute towards a decrease in computational effort. Table (3.6) below summarizes the execution times for the two sets of simulations.

Table 3.6: Comparison of Bayesian Updating Algorithms: Execution Times

Bayesian Updating Algorithm	100 Runs	1000 Runs
Theoretical	6.432 seconds	59.057 seconds
Approximate	4.295 seconds	58.888 seconds

These differences are well within the normal bounds of variation in execution time even for the same algorithm. That is, the approximate Bayesian estimator does not materially affect the execution time of the simulations.

This can be explained by the fact that, even though the approximate Bayesian estimator requires a single call to the exponential function, the formulae in equations (3.93) through (3.96) are used for the calculation of the unnormalized conditional probabilities. In order to calculate the normalized conditional probabilities, both $\tilde{p}_{\overline{\mu}}, t_{n+1}$ and $\tilde{p}_{\underline{\mu}}, t_{n+1}$ need to be calculated, and since the emission probabilities in these two cases are different, the exponential function needs to be called twice regardless. Consequently, since there is no significant gain in efficiency by using the approximate Bayesian estimators, we shall use exclusively equations (3.86) through (3.89) for the purposes of calculating recursive unnormalized conditional probabilities.

The estimation of conditional state probabilities demonstrated above is necessary in order to estimate fundamental values for each asset in the market. Recall equation (3.20). There, we specified how fundamental values under certainty are to be computed. Certainty in the context of our model refers to information about the underlying state of each industry. Full information is interpreted as certainty, while partial information is interpreted as uncertainty. There are two possible states, but as discussed in the previous sections, they remain unobserved and must be estimated from observable economic data. Now that we have provided a means to estimate conditional state probabilities, we can give an expression for fundamental values under uncertainty as well.

Denote by $\underline{FB}_{t_n}$ the fundamental value of an asset under certainty at time t_n, corresponding to the $\underline{\mu}$ state of the industry, and by $\overline{FB}_{t_n}$ the fundamental value under certainty, corresponding to the $\overline{\mu}$ regime. Then, the fundamental value of risky asset k at time t_n under uncertainty is given by:

$$FV_{k,t_n} = p_{t_n}\overline{FB}_{k,t_n} + (1 - p_{t_n})\,\underline{FB}_{k,t_n}, \tag{3.97}$$

where p_{t_n} is the normalized conditional probability of being in state $\overline{\mu}$ given in equation (3.85).

3.4.2.14 Summary

In this section, we considered in detail the issue of estimating fundamental values for the risky assets in the economy. To this end, we began by specifying expressions for the fundamental values under the assumption of full information about the state of each industry. These were obtained by applying Monte Carlo techniques to the standard discounted dividends methodology. In a situation like ours, where there are only two possible regimes, these fundamental values under certainty can be interpreted as boundaries for the estimated fundamental values under the assumption of partial information about the state process. That is, depending on the estimated probabilities of being in each of the two regimes, the estimated fundamental value under uncertainty will be somewhere between the two boundaries set by the fundamental values under certainty.

We then proceeded to give a review of the relevant theory of state estimation in hidden Markov models. In order to illustrate the relevant concepts and techniques, we started with a simplified example with discrete-range states and observations. The primary methodology used in solving the estimation problem was the reference probability method, which was implemented by means of a change of measure and the conditional Bayes theorem. The results from these sections were subsequently extended to the case of particular interest for our purposes: discrete-range states and continuous-state observations. No major difference in results or their interpretation was observed in the case of continuous-range observations.

We concluded this section with a discussion on several issues arising from the practical application of the presented state estimation algorithms. Two alternative Bayesian estimators were presented and justified. A simulation experiment was carried out with the objectives of verifying the equivalence of the two algorithms and comparing their computational complexity. While it was shown that the two algorithms produce equivalent results for all practical purposes, it was decided to adopt the theoretical formulation for a Bayesian estimator, since no significant gains in computational efficiency were observed when using the approximate version. Finally, an expression for the estimation of fundamental values under uncertainty was provided. Fundamental values are a vital ingredient for the specification of investor behaviour within our model. We now proceed to discuss in more detail how agents make their investment decision in out stylized model of financial market dynamics.

3.5 Investment Strategies

In this section we discuss in more detail another important building block of our model: how agents make investment decisions. We depart from the assumption of homogeneous beliefs imposed in the framework of mean-variance optimization, and instead allow the various agents populating the economy to pursue behaviour consistent with their own unique expectations. In other words, each type of investor is fully characterized by their unique preferences, which lead to specific behaviour in the market. Since the agents' investment decisions are not obtained as the solution of a utility maximization problem, but are taken as model primitives, we need to carefully specify the different types of investment behaviour, as well as the assumptions and motivation behind each such formulation.

To this end, we use the literature on boundedly rational heterogeneous agents as a starting point. We take the most widespread investment strategies as a basis, and extend the analysis by considering several new types of behaviour, which are well suited to modeling investment within the framework of institutional constraints.

Generally, there exist two wide strands of research regarding the specification of investment behaviour in the agent-based literature. One approach, encountered for example in Palmer et al. (1994), Palmer et al. (1999), and Ehrentreich (2008), takes a very unrestricted, evolutionary perspective on investment behaviour and allows agent behaviour to evolve freely by means of selection and mutation. The other approach, prevalent in works such as Chiarella et al. (2007), Chiarella et al. (2009), Lux (2009), Hommes & Wagener (2009), and Gaunersdorfer (2000), is to classify investment behaviour in several groups and examine the market dynamics and stylized facts generated by them, the switching of investors between the groups, as well as steady states or limit cycles in investment behaviour. The specific problem we are examining in this piece of research merits a slight departure from this approach.

In order to examine the robustness of different types of investment behaviour in an institutional framework with minimum guaranteed liabilities and macroeconomic shocks, we also classify investment strategies into different groups, but do not allow switching between them. In this way, we achieve wealth-driven selection of the most robust strategies, since the unsuccessful ones will be driven out of the market. We specify four broad groups of investment behaviour. In later chapters we also examine the robustness of our results by comparing them to two popular benchmark strategies. We keep to the types of a fundamentalist and a trend follower (chartist), prevalent in the agent-based literature, but also add two investment strategies based on dividend yields. The extension to the model discussed in chapter five, adds the strategies of naïve diversification and myopic mean-variance optimization as benchmarks. All investment strategies explain how agents make their decisions to invest in risky assets. The existence of a risk-free asset in our model adds another dimension to investment behaviour. Investments in the risk-free asset are not specified by means of an explicit strategy, but follow as a consequence of the assumptions of budget exhaustion and full diversification, once the agents have decided on their allocations to the risky assets.

We begin by discussing the two dividend yield strategies. The motivation behind specifying such types of investment behaviour is that fundamental information, such as dividend yields, price-earnings ratios, and return on equity, are frequently used by institutional investors in order to evaluate their potential investments. Furthermore, the regulatory framework in many countries requires institutional investors to invest only in mature, investment-grade companies, which tend to pay relatively stable dividends, and hence there is no real danger that a strategy specified in terms of dividends fails to be well-defined because of a dividend policy that does not pay out at least a portion of the company's earnings as dividends.

The first investor type we specify in our model will be referred to as "naïve dividend yield investor". This agent invests in the risky assets by simply calculating a dividend yield, subject to an appropriate normalization. Denote the portion of the naïve dividend yield investor's wealth to be invested in risky asset k at time t_n by λ_{k,t_n}^{ND}. Then, λ_{k,t_n}^{ND} will be given by:

$$\lambda_{k,t_n}^{ND} = \frac{\frac{\delta_{k,t_n}(t_n - t_{n-1})}{S_{k,t_{n-1}}}}{\sum_{k=1}^{K} \frac{\delta_{k,t_n}(t_n - t_{n-1})}{S_{k,t_{n-1}}} + r\,(t_n - t_{n-1})}, \tag{3.98}$$

where δ_{k,t_n} denotes the dividend intensity paid by asset k between times t_{n-1} and t_n, whereas $S_{k,t_{n-1}}$ is the market price of asset k at time t_{n-1}. The investment strategy λ_{k,t_n}^{ND} is calculated at time t_n, just before the new share price is set that will clear the market. Recall that the order by which trading takes place begins with formulating investment decisions, setting a market clearing price, and then transacting at that price to rebalance the portfolio.

In practice it would be possible to formulate a dividend yield investment decision at time t_{n-1} as companies usually announce their dividends a reasonable amount of time before they pay them. However, within the confines of our simulation study the dividend generating processes that we use do not allow the investor to know the t_n dividend values until the process actually reaches this point in time. Also note, that the normalizing factor in equation (3.98) includes the term $r\,(t_n - t_{n-1})$. This term ensures that the assumption of full diversification will not be violated: i.e. the allocation to asset k will never constitute the investor's entire wealth endowment. Therefore, there will always be enough funds available to invest in the risk-free asset as well. Both the dividend intensity and the interest rate on risk-free investment are scaled according to the length of the time periods.

Using the same notation as before, denote the allocation of the naïve dividend yield investor to the risk-free asset at time t_n by λ_{0,t_n}^{ND}. Then, by the assumption of budget exhaustion, we have:

$$\lambda_{0,t_n}^{ND} = 1 - \sum_{k=1}^{K} \lambda_{k,t_n}^{ND} = \frac{r\,(t_n - t_{n-1})}{\sum_{k=1}^{K} \frac{\delta_{k,t_n}(t_n - t_{n-1})}{S_{k,t_{n-1}}} + r\,(t_n - t_{n-1})}. \tag{3.99}$$

The naïve dividend yield strategy tends to be quite volatile, tracking closely the volatility of the dividends. In times of recession it naturally protects the investor by decreasing their allocation to the risky assets as dividends decline, however, this is somewhat offset by the asset prices, which also tend to decline during this stage of the business cycle. Since the decline in dividends during recessions happens rapidly, but it takes a certain amount of time for investors to reduce their allocations to the risky assets, and thus depress their prices, the protective benefits of the naïve dividend yield strategy are felt most strongly during the initial stages of recessions.

In order to examine the question of whether there are any benefits of being a more sophisticated and informed investor, we also provide a more complex version of the simple dividend yield strategy above. This is our second dividend yield strategy, which shall be referred to as "sophisticated dividend yield" strategy. This investment approach tries to eliminate some of the noise, inherent in using market data, by estimating an expected average dividend yield.

Denote by λ_{k,t_n}^{SD} the proportion of the sophisticated dividend yield investor's wealth to be invested in risky asset k at time t_n. We specify λ_{k,t_n}^{SD} as follows:

$$\lambda_{k,t_n}^{SD} = \frac{\frac{\hat{\mu}_{k,t_n}(t_n - t_{n-1})}{S_{k,t_{n-1}}}}{\sum_{k=1}^{K} \frac{\hat{\mu}_{k,t_n}(t_n - t_{n-1})}{S_{k,t_{n-1}}} + r\,(t_n - t_{n-1})}, \tag{3.100}$$

where $\hat{\mu}_{k,t_n}$ is the estimated expected dividend: $\hat{\mu}_{k,t_n} = p_{t_n}\overline{\mu}_k + (1 - p_{t_n})\,\underline{\mu}_k$. Equation (3.100) is normalized in the same way as (3.98) in order not to violate our assumption of full diversification.

The allocation of the sophisticated dividend yield investor to the risk-free asset is given in the same way as above:

$$\lambda_{0,t_n}^{SD} = 1 - \sum_{k=1}^{K} \lambda_{k,t_n}^{SD} = \frac{r\,(t_n - t_{n-1})}{\sum_{k=1}^{K} \frac{\hat{\mu}_{k,t_n}(t_n - t_{n-1})}{S_{k,t_{n-1}}} + r\,(t_n - t_{n-1})}. \tag{3.101}$$

Compared to the naïve dividend yield strategy, the sophisticated dividend yield strategy is less volatile and less affected by market data. This brings more stability in the investment behaviour and helps to reduce transaction costs, but a potential weakness of this strategy is that it is somewhat slower to respond to changing market conditions. Of course, if the regime estimation is good enough, these changing market conditions will have already been forecast with a reasonable degree of accuracy.

Next, we discuss the popular strategies of a fundamentalist and a trend follower. The agent-based literature provides a variety of different specifications for this type of investment behaviour. For instance, Lux (2009) uses the hyperbolic tangent function *tanh* in order to model the switching of agents between the groups of pessimists and optimists, which affects their investment behaviour. The same function is used by Gaunersdorfer (2000) and Chiarella et al. (2009) in order to model switching between the strategies of a fundamentalist and a trend follower.

This formulation, of course, is a byproduct of the specific discrete choice probabilities that these authors use. Namely, social interactions are modeled by means of a multinomial logit model, also known as Gibbs' probabilities, of the form: $\frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}}$ (see Hommes & Wagener (2009)). Even though we do not have this type of switching between strategies in our model, we would still like to model the fundamentalist and trend follower strategies in terms of switching between the risky assets and the risk-free asset. Therefore, we use a similar formulation. However, as was discussed above, additional complexity is added by the presence of the risk-free asset. As a consequence, the calculation of the investment strategies of the fundamentalist and the trend follower will be specified as a four-stage process.

We will give the investment strategy of the fundamentalist first:

- Step 1:

$$\hat{\lambda}^F_{k,t_n} = \frac{\pi - \left(\frac{\pi}{2} + arctan\left(\alpha^F\left(S_{k,t_{n-1-T^F}} - FV_{k,t_{n-1-T^F}}\right)\right)\right)}{\pi}. \tag{3.102}$$

This step calculates an unnormalized value for the allocation of the fundamentalist to risky asset k. The inverse tangent function is used instead of the hyperbolic tangent. The term α^F is a scaling factor, which determines the strength of the fundamentalist's reaction to perceived over- or under-valuation of the corresponding asset. When the asset is perceived to be fairly valued, the fundamentalist will divide their money equally between the risky and the risk-free asset. When there is a perceived under-valuation (i.e. $S_{k,t_{n-1-T^F}} < FV_{k,t_{n-1-T^F}}$), the fundamentalist will increase their allocation to the risky asset and vice versa. This behaviour is depicted in figure (3.7). In the above formulation T^F refers to the length of the lookback period that the agent uses in decision-making. For the fundamentalist it is usually set equal to zero, as this strategy relies on detecting misvaluations in the present, without the need to refer back to past price history.

The formulation outlined in equation (3.102) would have been sufficient of we had to model a single asset and there was no assumption of full diversification. In the case of multiple assets, however, additional normalization is required. This is shown in the steps below:

- Step 2:

$$\hat{\hat{\lambda}}^F_{k,t_n} = \frac{\hat{\lambda}^F_{k,t_n}}{\sum_{k=1}^{K} \hat{\lambda}^F_{k,t_n}}. \tag{3.103}$$

Equation (3.103) is the first step in dealing with multiple assets. Once the normalized preliminary allocations $\hat{\hat{\lambda}}^F_{k,t_n}$ have been calculated, the next step is to use these normalized preliminary allocations in order to examine the degree of over- or under-valuation of the whole portfolio of risky assets, on condition

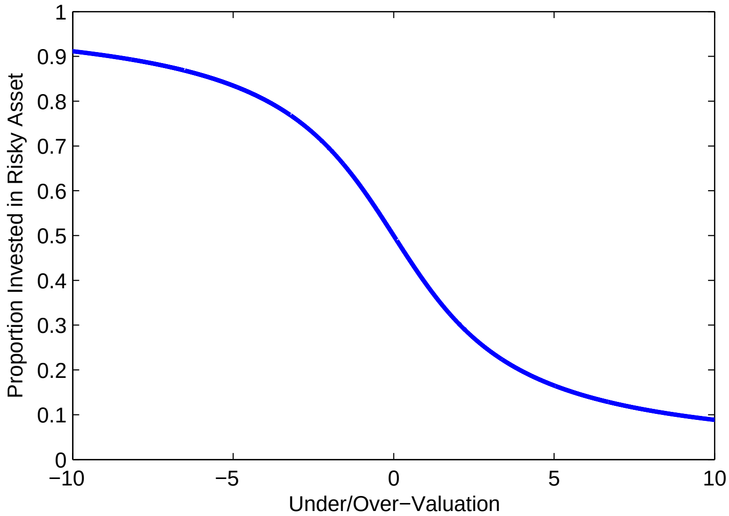


Figure 3.7: Investment Strategy based on Fundamental Value: Illustration

that the investor allocates their wealth according to $\hat{\hat{\lambda}}^F_{k,t_n}$. This is necessary in order to calculate an allocation to the risk-free asset. The logic here is the same as the one shown in figure (3.7), but in this case the fundamentalist switches between a portfolio of risky assets and the risk-free asset, in accordance to the misvaluation of the whole portfolio, instead of a single risky asset:

- Step 3:

$$\lambda^F_{0,t_n} = 1 - \frac{\pi - \left(\frac{\pi}{2} + arctan\left(\alpha^F\left(\sum_{k=1}^{K}\frac{\hat{\hat{\lambda}}^F_{k,t_n}V^F_{t_{n-1}}}{S_{k,t_{n-1}}}\left(S_{k,t_{n-1-T^F}} - FV_{k,t_{n-1-T^F}}\right)\right)\right)\right)}{\pi}. \tag{3.104}$$

The logic of equation (3.104) is similar to the well-known two fund separation theorem. The degree of misvaluation of the whole portfolio of risky assets determines the allocation to the risk-free asset. After making an allocation to the risk-free asset, the fundamentalist has the rest of his funds available to invest in risky assets. This is done according to how over- or under-valued each of the risky assets is estimated to be. That is, the preliminary risky allocations $\hat{\hat{\lambda}}^F_{k,t_n}$ are weighted by the amount of funds available for risky investments. In other words, the fundamentalist's allocation to risky asset k at time t_n is given by:

- Step 4:

$$\lambda^F_{k,t_n} = \left(1 - \lambda^F_{0,t_n}\right)\hat{\hat{\lambda}}^F_{k,t_n}. \tag{3.105}$$

An alternative specification, satisfying all requirements for the investment behaviour of a fundamentalist, may also be provided by substituting the function $\frac{(\alpha x)^2}{\beta+(\alpha x)^2}$ for the inverse tangent function $arctan$ and appropriately scaling it. Then, the equivalent to equation (3.102), for example, would be given by:

$$\hat{\lambda}^F_{k,t_n} = \frac{2 - \left(1 + sgn\left(S_{k,t_{n-1-T^F}} - FV_{k,t_{n-1-T^F}}\right) \frac{\left(\alpha\left(S_{k,t_{n-1-T^F}} - FV_{k,t_{n-1-T^F}}\right)\right)^2}{\beta + \left(\alpha\left(S_{k,t_{n-1-T^F}} - FV_{k,t_{n-1-T^F}}\right)\right)^2}\right)}{2}, \tag{3.106}$$

where, α and β are scaling parameters and

$$sgn(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

The investment behaviour of the fundamentalist according to equation (3.106) is illustrated in figure (3.8). This specification results in behaviour, which

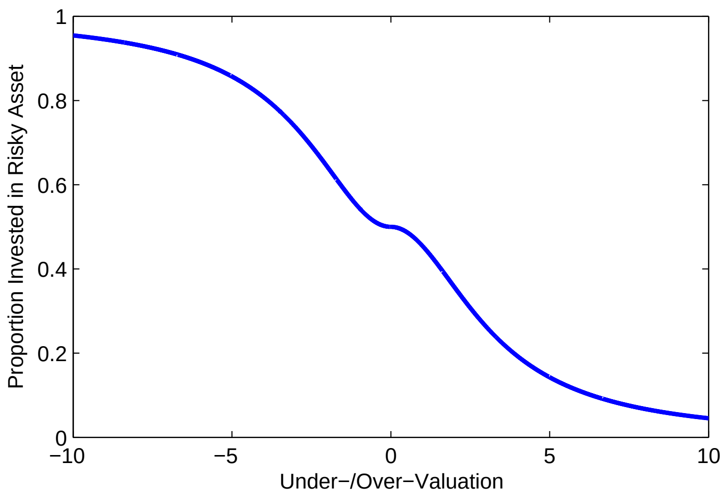


Figure 3.8: Alternative Fundamental Investment Strategy: Illustration

is quite similar to the one in equation (3.102). Since there are no specific advantages in using formulation (3.106) instead of (3.102), we will use the specification in equation (3.102) in order to model the investment behaviour of the fundamentalist, since it is smoother and slightly more responsive in the interval where the risky asset is fairly valued.

The fundamental investment strategy tends to be the most stable of the four. This is, of course, conditional on the sensitivity parameter α, but generally this strategy is the least affected by market fluctuations, since both fundamental values and prices are aggregate quantities, and hence change at a slower pace than dividends or the conditional probabilities of the regimes of each industry. Additionally, the fundamentalist has the greatest impact on market efficiency, since his trading behaviour naturally forces prices to converge to fundamental values.

The other strategy that we borrow from the agent-based literature is that of a trend follower, or a chartist as it is sometimes referred to. This strategy is a proxy for short-term speculative behaviour, which is driven purely by asset price momentum, and is the complete opposite of the fundamental investment strategy, in the sense that it reinforces movement of the asset price in one direction for a significant amount of time and contributes to the formation of asset price bubbles. When the market is dominated by this type of behaviour, this is usually interpreted as the primary explanation for some stylized facts, such as the fat tails of returns distributions and volatility clustering. Unlike the fundamentalist strategy discussed above, trend following usually has a destabilizing effect on financial markets and contributes towards inefficiencies, quantified in terms of deviations from fundamental values.

We specify the behaviour of a trend follower by means of a similar four stage procedure as above, with an additional consideration. Any kind of trend following requires a certain minimum amount of price history to be available. This requirement is obviously not satisfied in the initial stages of a simulation. Therefore, whenever there is no sufficient price history available, a trend follower will behave as a fundamentalist. Once the required minimum amount of price history has been reached, then the chartist will switch to trend following. In other words, the equivalent to equation (3.102) in the case of a trend follower is given by:

- Step 1:

$$\hat{\lambda}^{TF}_{k,t_n} = \begin{cases} \dfrac{\pi - \left(\frac{\pi}{2} + arctan\left(\alpha^F\left(S_{k,t_{n-1-T^F}} - FV_{k,t_{n-1-T^F}}\right)\right)\right)}{\pi}, & t_{n-1} < T^{TF} \\ \dfrac{\frac{\pi}{2} + arctan\left(\alpha^{TF}\left(ln\left(\frac{S_{k,t_{n-1}}}{S_{k,t_{n-1-T^{TF}}}}\right)\right)\right)}{\pi}, & t_{n-1} \geq T^{TF}, \end{cases} \tag{3.107}$$

where T^{TF} denotes the length of the past price history period that the trend follower uses in his decision-making, and α^{TF} is a scaling parameter, which determines how sensitive the trend follower is to price momentum over the indicated time period T^{TF}. Note that the fundamentalist and trend follower have different sensitivity parameters. This is so because of the different behaviour that is expected from these two groups of investors, as well as the different size of the arguments to the *arctan* function. While fairly large deviations of the price from fundamental value may occur, the return over the last T^{TF} periods – $ln\left(\frac{S_{k,t_{n-1}}}{S_{k,t_{n-1-T^{TF}}}}\right)$ – rarely reaches such large values. Therefore, the trend follower needs to react in a much stronger fashion to smaller *arctan* argument values. The trend follower strategy is also very sensitive to the choice of a lookback period T^{TF}. A graphical illustration of the investment behaviour of the trend follower is provided in figure (3.9).

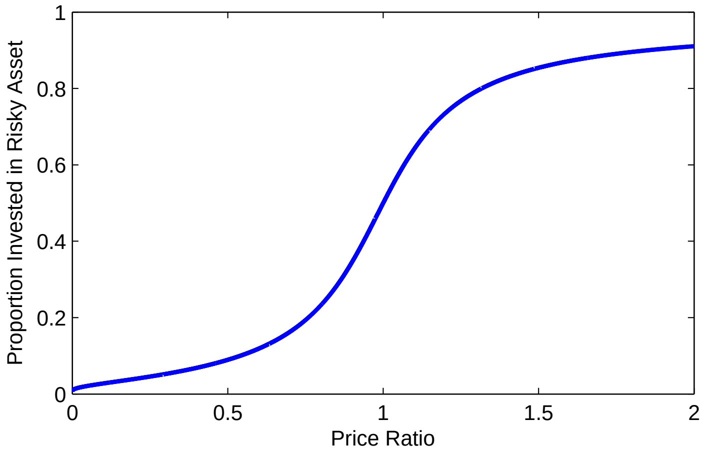


Figure 3.9: Trend Following Investment Strategy: Illustration

The preliminary normalized risky asset allocations for the trend follower are specified in the same way as for the fundamentalist:

- Step 2:

$$\hat{\hat{\lambda}}_{k,t_n}^{TF} = \frac{\hat{\lambda}_{k,t_n}^{TF}}{\sum_{k=1}^{K} \hat{\lambda}_{k,t_n}^{TF}}. \tag{3.108}$$

Once these preliminary normalized allocations have been computed, they can be used in order to compute the trend follower's investment in the risk-free asset in a way, similar to step 3 in the procedure for the calculation of the investment strategies of the fundamentalist. As before, implementing a trend following strategy is conditional on there being a sufficient amount of price history.

- Step 3:

$$\lambda_{0,t_n}^{TF} = 1 - \begin{cases} \dfrac{\pi - \left(\frac{\pi}{2} + arctan\left(\alpha^F\left(\sum_{k=1}^{K} \frac{\hat{\hat{\lambda}}_{k,t_n}^{F} V_{t_{n-1}}^{F}}{S_{k,t_{n-1}}}\left(S_{k,t_{n-1-T^F}} - FV_{k,t_{n-1-T^F}}\right)\right)\right)\right)}{\pi} \\ \dfrac{\frac{\pi}{2} + arctan\left(\alpha^{TF}\left(\sum_{k=1}^{K} \frac{\hat{\hat{\lambda}}_{k,t_n}^{TF} V_{t_{n-1}}^{TF}}{S_{k,t_{n-1}}} ln\left(\frac{S_{k,t_{n-1}}}{S_{k,t_{n-1-T^{TF}}}}\right)\right)\right)}{\pi}, \quad t_{n-1} \geq T^{TF}. \end{cases} \tag{3.109}$$

While the second expression in equation (3.109) technically satisfies the requirement for full diversification, it is not so easy to justify it theoretically as was the case with the fundamentalist, where such a normalization was intuitively motivated with the help of the two fund separation theorem.

Once the investment in the risk-free asset has been computed, the preliminary normalized allocations $\hat{\hat{\lambda}}^{TF}_{k,t_n}$ are weighted by the amount of funds available for risky investment in order to obtain the trend follower's allocations to the risky assets:

- Step 4:
$$\lambda^{TF}_{k,t_n} = \left(1 - \lambda^{TF}_{0,t_n}\right) \hat{\hat{\lambda}}^{TF}_{k,t_n}. \tag{3.110}$$

The trend following strategy reacts to changing market conditions the quickest when compared to the other three strategies discussed above. Most of the time the trend following strategy decreases market efficiency, not only because it pushes prices away from fundamental values, but also because it contributes a great deal of noise to the market. Nonetheless, in situations when such overreactions to changing market condition are warranted, this type of investment behaviour may result in sizeable rewards for the investor.

In this section we presented an important building block for our agent-based model of financial market dynamics. Namely, the four main types investment behaviour were described and a discussion was provided, which explained in detail how each group of agents makes their investment decisions. The types of investment behaviour we have selected are a mix between the two most popular trading strategies in the agent-based literature, as well as strategies based on fundamental market information, such as dividend yields, which are popular among regulated institutional investors.

In the latter group, we discussed the strategies of a naïve and sophisticated dividend yield investors. The former invests simply on the basis of dividend yields, subject to appropriate normalization. The benefits of this strategy, besides being simple to formulate and execute, are that it provides short-term protection against adverse market movements. This is because recessionary periods manifest themselves in the form of decreased dividends much quicker than deflation of asset prices, and this leads to reduced allocations to risky assets in the beginning of recessionary periods.

Sophisticated dividend yield investors, on the other hand, are not as quick to react to signs of recessionary periods, such as declining dividend income, but attempt to estimate conditional probabilities of a particular industry being in recession. The conditional probabilities of both regimes are then used to estimate average dividend yields, on the basis of which investment decisions are taken. Compared to the naïve dividend yield strategy, this group of investors are capable of learning about the states of each industry and their asset allocations tend not to be as volatile, owing to the smoother estimated average dividend yields.

The second group of investment behaviour we have selected are the well-known agent types of a fundamentalist and a trend-follower. These trading strategies are characterised by switching between risky and risk-free assets based either on perceived misvaluation or price momentum. The fundamentalist type increase their allocation to risky assets when they estimate the asset price to be less than an asset's fundamental value and vice versa. This type of behaviour is usually quite stable and contributes to the existence of efficient markets, since when fundamentalists dominate the market, asset prices converge to fundamental values through the investment activities of the fundamentalists. Conversely, trend followers invest on the basis of asset price momentum, which, if sustained, leads to a divergence between prices and fundamental values, as well as the formation of asset price bubbles.

The collection of investment strategies outlined in this section are conceptually simple. Agents make their decisions based on a set of rules, which can be either technical or fundamental in nature. Since investment strategies are considered model primitives and there is an almost infinite number of possible types of investment behaviour, we make no attempt to generalise and prescribe a single strategy, which is optimal under all circumstances. Rather, we try to explain which features of the different types of investment behaviour are selected by the market under varying scenarios, particularly during recessionary periods and asset price deflation.

In subsequent chapters we will compare the results from our model with equivalent results for other popular types of investment behaviour, such as the Kelly rule, which has been proven to be a superior long-term strategy in the absence of minimum consumption constraints (Thorp (2006)). We also extend the model by including two more popular types of investment behaviour to act as a benchmark. Namely, we include the strategies of naïve diversification and mean-variance optimisation. We discuss some difficulties in the practical implementation of these strategies within the framework of our model's assumptions and investigate whether the inclusion of these additional types of investment behaviour makes a difference to our initial market selection results. We wrap up the current chapter with a short discussion concerning the software implementation of our simulation model.

3.6 Implementation

In this section we present an overview of the model implementation in the C++ programming language. The full program code is provided in the accompanying CD to this thesis.

The implementation consists of several separate building blocks and supports both object-orientation and generic design. The building blocks are as follows: a header file containing the implementation of necessary classes, a header file containing both the definition and the implementation of the main functionality of the program, an interface file for communicating with the user and obtaining model parameters, a file containing the implementation of random number generating functions for the uniform, standard normal and exponential distributions, as well as a separate file containing ancillary functionality such as error checking. We have also used the ALGLIB open source library of functions (Bochkanov 2012) for some of the common mathematical operations like spline interpolation and solving systems of simultaneous linear equations. We now proceed to discuss each of the above in more detail.

3.6.1 Classes

The first building block of our implementation is the creation of the necessary classes to support object-orientation. This is done in a separate header file. For our purposes only a single class turned out to be sufficient. We defined and implemented a dynamic array class for the purposes of easier manipulation of two-dimensional arrays.

This class was implemented as a template wrapper around vectors from the corresponding class from the standard template library (STL). Thus, each row is created using an STL vector, whose every element contains another STL vector to represent the second dimension of the array. Appropriate overloading was also implemented so as to allow for easy individual element access and for other necessary member functions, such as vector length, for instance. The class is operational with both constant and non-constant objects.

3.6.2 Core Functionality

The core functionality of the simulation software was implemented in a separate header file containing six major functions, as well as a number of ancillary functions, each designed to handle a specific simulation task. Both the declaration and the definition of these functions were implemented in the same header file because of the generic programming capabilities provided by the template functionality of C++. The transfer of inputs and outputs between the different functions, whenever necessary, is accomplished by means of STL vectors containing instances of the dynamic array class discussed above. In the following subsections we proceed to quickly outline the most important features of each of the major functions necessary for the simulation.

3.6.2.1 Regime Switching Function

The task of the regime switching function is to simulate the mean-reverting dividend levels for each state of an industry. This is accomplished by taking in user inputs about the desired mean-reverting dividend values and discretizing time into smaller time-steps.

As discussed in the previous sections, we model the waiting times between regime switches by means of an exponential distribution. During each time step, a test for a regime switch is performed by generating a uniformly distributed random variable and checking if it is greater than $e^{-\lambda dt}$, where dt is the size of the time step in the discretization and λ is the parameter of the exponential distribution. The latter changes according to the particular regime, so that recessions last for a shorter duration than normal economic conditions as discussed above. The desired mean-reverting values in each regime are then stored and, if need be, passed to other functions, such as the dividend function discussed below. The above functionality is implemented in the C++ function “jumps” presented in the accompanying CD.

3.6.2.2 Dividend Function

Having generated the mean-reverting values for the corresponding regimes of an industry in the previously discussed function, the dividend function simulates one of the two dividend generating processes given in equations (3.15) and (3.18). The choice of the particular process to be used is left to the user.

The same discretization of time as in the regime switching function is used, although because of considerations that will become evident in the next chapter, the investment horizon determined by the user is slightly extended with an arbitrary number of additional time steps, so that the simulated dividend process exceeds the user-defined investment horizon.

The dividend function is related to two additional ancillary functions, whose purpose is to enhance user control over the simulation process. Firstly, the user is given the capability to directly control the evolution of the regimes of an industry by explicitly setting the time of the regime switches. This is accomplished by a separate function and deployed by a user-defined dummy variable. If a specific scenario for the regime switches is not needed, then the regime switching function is called and regime switches are generated according to the relevant probability laws.

Secondly, the user is also given control over the randomness in the dividend generating processes. This is achieved by giving the user the options to save and later load specific realizations of the Brownian increments used in the dividend function. Thus, the impact of changes in the regimes of an industry can be analyzed and the conclusions reached in this way will not be influenced by the short-term random dividend fluctuations. If the user specifies that the saved Brownian increments are to be loaded and used, then new Brownian increments are not generated during the execution of this function. If, however, a file containing the necessary Brownian increments is not found in the corresponding directory, then the user input is overridden and new random fluctuations will be generated.

Depending on the user choice, one of the discrete versions of the dividend generating processes (3.15) or (3.18) is then simulated and stored, ready to be passed on to another function that may need dividend realizations as an input. The functionality described above is implemented in the function called "milstein" in the C++ code below.

3.6.2.3 Fundamental Values Under Certainty

Since our model requires the estimation of fundamental values, we begin by designing a function for the calculation of fundamental values under certainty. Certainty in this context refers to certainty about which regime an industry is in. This is by far the most computationally intensive task in our model. If we treat fundamental values under certainty as a function of dividends, then the calculation of just a single point in the fundamental value curve under certainty requires a Monte Carlo simulation with a large sample size. We have therefore dedicated a separate function to handle this complex task.

To begin the fundamental value calculation, the user is given the option to specify a range of starting values for the dividend generating processes in equations (3.15) and (3.18). The fundamental value curves under certainty for each risky asset that will be obtained as the output of this routine will be functions of these starting dividend values – one curve for the recession regime and another one for normal economic conditions.

Depending on the parameters for the main wealth dynamics simulation that one uses, it is possible to reduce the run time of this routine by selecting an appropriate range of dividend values, such that it covers, with some probability, the range of dividend values that might occur when equations (3.15) and (3.18) are simulated, but is not excessively broad.

Specifically, the lower boundary for the range should always be zero, since dividends can never go below this value, whereas the upper boundary can be determined by running the dividend function multiple times with the particular starting dividend parameters that the user wishes to use and determining a value, which is unlikely to be surpassed in magnitude by the different dividend realizations. While this value needs to be large enough to cover all possible dividend scenarios, it must not be excessively high, as this will lead to a lot of wasted computing time. To give some perspective on the possible range of values in this step, for the case of two risky assets with starting dividend values $D_0^1 = 1.25$ and $D_0^2 = 1.75$ an upper boundary of 20 proves to be more than sufficient.

Once the range for the starting dividend values has been obtained, the number of points in the fundamental value curves under certainty that are actually calculated by means of Monte Carlo simulations is determined by another user-defined parameter. One option is to merely space these points linearly across the starting dividend values by dividing the range of starting dividends by the desired number of fundamental value points. While nearly any number of points will work well for the mean-reverting square root process in equation (3.15), this is not so for the case of the geometric Ornstein-Uhlenbeck process (3.18).

This phenomenon occurs because of two main reasons. Firstly, the general shape of the fundamental value curves under certainty is determined by the specific dividend generating process in use. If the mean-reverting square root process (3.15) is used, then the fundamental value curve under certainty is simply a linear function of the starting dividend values. Whenever the geometric Ornstein-Uhlenbeck process (3.18) is used to simulate dividend realizations, however, the fundamental value curve under certainty will be a strictly increasing, concave function of the starting dividend values.

Secondly, in order to keep computing time at a manageable level, we obtain only a limited number of points in the fundamental value curves and then extrapolate the rest by fitting a function through them. A modeler has two main choices in this respect – one can either opt for a simpler, more analytically tractable function, while sacrificing some accuracy, or a more complex function can be used, which does go through all of the fundamental value points, but is not as easy to manipulate. In our case, we have made the decision to use the latter alternative. The gaps between the fundamental value points were filled by means of spline interpolation because of their flexibility and high R^2.

Unless simple linear splines are used, any other form of splines that requires the calculation of derivatives might cause some problems with the fundamental value curve under certainty when the geometric Ornstein-Uhlenbeck process is used, if the spacing of the fundamental value points is not carefully determined. This is because the fundamental value curve under certainty associated with this dividend generating process has two distinct parts – a steeper and a flatter part. The transition between the two may sometimes cause a certain inconsistency when splines are used to extrapolate between points in this region of the curve.

Because of the changing slopes in this region, if the fundamental value points are spaced too wide apart, fitting a spline through them might cause a violation of the monotonicity of the fundamental value curve under uncertainty. This outcome will be graphically illustrated in the subsequent chapter and is highly undesirable since it violates the well-established economic principle that fundamental values must be increasing functions of dividends.

The situation in the previous paragraph may be averted by spacing the fundamental value points closer to each other and using simple linear splines to connect them. A significant drawback of this approach is that in order for a smooth fundamental value curve under certainty to be obtained, one would need to use very close spacing between the individual points. Therefore, if these fundamental value points are linearly spaced across the range of starting dividend values, then this would entail the calculation of a large number of them, which would significantly extend the necessary computing time.

Another alternative is to space the fundamental value points not linearly across the range of starting dividends, but logarithmically. In this way the fundamental value points in the initial steeper part of the curve will be situated closer to each other, which will take care of the monotonicity problem in the transition region, but in the same time they will be spaced further away from each other in the flatter region of the curve, where closer spacing is not required. This would provide an optimal tradeoff between computational effort and accuracy.

Once the number of fundamental value points and their spacing has been determined, arrays with appropriate dimensions are set up so as to hold various results and the dividend function is called multiple times for each regime of an industry, with each call returning batches of 1000 realizations for each risky asset. Then, for each realization the sum of the discounted dividends is calculated and stored. Finally, each fundamental value point under certainty is obtained by taking the sample mean of the sum of the discounted dividend realizations. In order to check the quality of the Monte Carlo simulations and to gauge if the sample size is sufficient, confidence intervals around each fundamental value point are also calculated.

The next step in the calculation of the fundamental value curves under certainty is to fill in the blank spaces between the calculated fundamental points.

We do this by means of an Akima spline interpolation and utilize ALGLIB's (Bochkanov (2012)) functionality in order to implement it. Since the resulting fitted function is not a real analytic function, such as the exponential function for instance, we store the entire output of this step. The ALGLIB library provides routines for the calculation of spline interpolation and stores the output internally as a special structure. We extended ALGLIB's functionality by amending some of its classes in order to allow writing the structure containing the outputs from the spline interpolation routines to a file. In this way we store the fundamental values under certainty for each of the risky assets under both regimes of an industry and using both dividend generating processes in equations (3.15) and (3.18).

The reason why we would like to store the outputs from this function to a file is the fact that due to the computational complexity of this routine and the large amount sample runs needed for each asset and for each dividend generating process, it can take over twenty-four hours of computing time to obtain the outputs from this function. Therefore, it is not practical to run this routine every time one needs to run the main wealth dynamics simulation that will be discussed below. By running this function only once and storing all the outputs, they can easily be read in and used multiple times for the calculation of fundamental values under uncertainty, and hence for the calculation of investment strategies in the main wealth dynamics simulation with almost no additional computational effort.

In order to make the outputs from this function relevant to a wide variety of possible scenarios, we run it four times with different discount factors – 5%, 10%, 15%, and 20% – to represent four different scenarios for the levels of the interest rates in the economy that may be applicable when performing the main wealth dynamics simulation. So, in total, for the case of two risky assets, two dividend generating processes, two regimes in each industry, and four different levels of interest rates, the output from this routine is a collection of thirty-two different files, each containing the fundamental value curves under certainty for each of the risky assets in the corresponding regimes of the industry. These files are stored in an appropriate directory and are only loaded as they become necessary for some of the functions that follow, without the need to run this computationally demanding function again. The functionality discussed above is implemented in the function named “funBounds” in the C++ code presented in the accompanying CD.

3.6.2.4 Regime Estimation and Fundamental Values Under Uncertainty

The fundamental value function under certainty discussed in the previous subsection is only the first step in appropriately estimating a fundamental value for each of the risky assets. Generally, agents lack explicit information regarding the regime that a particular industry is in at each point in time. The regime switches form a hidden Markov model, as was discussed in the previous sections, and additional uncertainty is contributed by the short-term random oscillations in the dividend realizations. Investors within the context of our model are aware of the functional form of the dividend generating processes and know the mean-reverting dividend values for each regime, but do not know which regime of the industry is currently operational, so they need to be able to compute probabilities of being in each of the regimes of an industry conditioned on the observed dividend realizations, as well as to be able to learn and update these probabilities as new dividend realizations become available.

This probability estimation and learning process is handled by a separate regime estimation function. The prior for the starting probabilities of being in each of the regimes is set to the expected value of an exponentially distributed random variable. In order to avoid any bias in the beginning periods of the regime estimation procedure, the starting regime for the dividend generating processes is picked randomly. Then, the dividend function is called in order to generate a dividend realization with the desired parameters starting from the randomly picked regime. The estimation of regime probabilities and their subsequent revisions is implemented in the form of the set of equations (3.86) through (3.89) or (3.93) through (3.96). Since it was previously established that there were no significant economies in computational effort resulting from using the set of equations (3.93) through (3.96), the formulation in equations (3.86) through (3.89) is used for the purposes of regime estimation and updating.

Once both the fundamental values under certainty and the conditional probabilities of each regime have been calculated, another function estimates the fundamental values under conditions of uncertainty about the underlying regime of each industry.

This procedure is conceptually quite simple: first it calls the regime estimation function in order to obtain the conditional regime probabilities, then, depending on the desired dividend generating process and discount rate, it loads the appropriate fundamental values under certainty from the collection of stored files already containing this information. Finally it estimates the fundamental values under uncertainty by applying the probability of being each regime to the corresponding fundamental value under certainty. The estimation of fundamental values is then complete and the output is ready to be used by the main wealth dynamics simulation. The implementation of the functionality described in this subsection can be found in functions "hmm" and "funValues" presented in the accompanying CD.

3.6.2.5 Wealth Dynamics Function

The main functionality necessary for the implementation of our model of financial market dynamics is the simulation of the evolution of the investors' wealth endowments given by equation (3.9). This is handled by the wealth dynamics function. It uses all the outputs from the previously described functions in some form, either directly or indirectly. It also implements the remaining building blocks of the model.

Time is again discretized in the same way as in the previous functions. At the beginning of each time step, a test for bankruptcies is performed. If it successful and there are at least two solvent investors remaining, the simulation continues. If a bankruptcy has been detected then the bankruptcy procedure discussed in the previous sections is implemented in order to handle any outstanding short positions.

Next, the consumption for the time period is determined. Depending on the size of the wealth endowment of each agent, consumption can be either proportional to wealth or the fixed threshold m. Since the inclusion of this additional constraint is one of the main contributions of our model, the outcome of this is carefully scrutinized and saved in a separate file. In this way, the behaviour of the wealth dynamics can be analyzed during two distinct periods – times when proportional consumption is used and times when the minimum threshold m must be used.

Since investment decisions in the context of our model take place at discrete points in time, and since asset allocations are decided in advance at the beginning of each time period, at the end of the time periods there is sufficient information in order to determine the prices of the risky assets. As was discussed in the previous sections, this happens endogenously. The price of the risk-free asset is taken as the numeraire. Once the investment proportions and the asset prices have been determined, the amount of shares held by each of the investors can be calculated.

The penultimate step in the wealth dynamics function is the computation of the investment proportions for the next time period. This happens by means of implementing equations (3.98) through (3.110). Since the determination of some of the investment strategies requires multiple intermediate calculations, the latter are broken down into separate ancillary functions, each designed to handle a specific step in the calculation process. This helps to locate early unintended investment behaviour and to signal potential problems with either the design of the trading strategies or the model parameters used in the simulation. As will be discussed in the next chapter, the parameters specifying the initial investment proportions at time t_0 are particularly difficult to set and deviations from their optimal values may sometimes cause the agents populating our model to exhibit undesirable investment behaviour.

Finally, once the allocations to all risky assets have been established, the budget constraint for each investor determines their allocations to the risk-free asset. With this, all the necessary components for the calculation of the wealth dynamics have been determined. After arranging them in the appropriate vector and matrix forms, the set of equations implied by (3.9) is solved. To this end, we use ALGLIB's functionality for the solution of systems of equations and avoid explicitly calculating the inverse matrix in (3.9).

The above sequence of steps is then repeated. The wealth dynamics simulation terminates in only two cases: either the end of the user-defined investment horizon has been reached, or a single solvent investor remains. In the latter case, the surviving investor type has accumulated all the wealth in the marketplace and has the full capacity to set asset prices. If the consumption parameters have been reasonably picked, so that consumption is not drastically higher than the income accruing to the investor, then the single survivor will never go bankrupt. This is a rather trivial situation and since no interesting analysis is possible at this point, the wealth dynamics simulation is discontinued.

Having outlined the structure of the core functionality of the simulation program, and having stressed some of the more important points, we now proceed with a discussion relating to some of the supporting functionality necessary for the proper implementation of the simulation program.

3.6.3 Pseudo-Random Number Generation

Pseudo-random number generation is an important part of the wealth dynamics simulation. Uncertainty is inherent in financial markets and the ability to simulate it efficiently is a vital prerequisite for making adequate investment decisions.

Our implementation of pseudo-random numbers follows closely the work by Marsaglia & Tsang (2000). Their ziggurat method was chosen because of its excellent efficiency, high-quality random numbers, and its ease of implementation. In the following paragraphs we provide a short description of the method and its implementation.

Usually, generating pseudo-random numbers from different distributions entails randomly picking points in the plain and accepting them if they fall within the probability density function of the required distribution. Otherwise, the point is rejected and a new randomly selected point is generated. The drawback of such an acceptance-rejection procedure is that it sacrifices efficiency. A lot of computational effort is wasted on rejected points, so the run time of such procedures can be significant. The ziggurat method is a special way of designing the acceptance-rejection procedure for classes of distributions with decreasing densities.

The two main strong points of the ziggurat algorithm are that firstly, it is very efficient in the sense that very few points will be rejected, which drastically speeds up random number generation, and secondly, in the vast majority of cases, the decision whether to reject or accept a point is very fast and simple, since it does not require the calculation of the probability density function of the desired distribution.

Both of these advantages are achieved by covering the decreasing density by a collection of simple structures, such as rectangles with the same area. An illustration with 7 covering rectangles and an unbounded base is presented in figure (3.10).

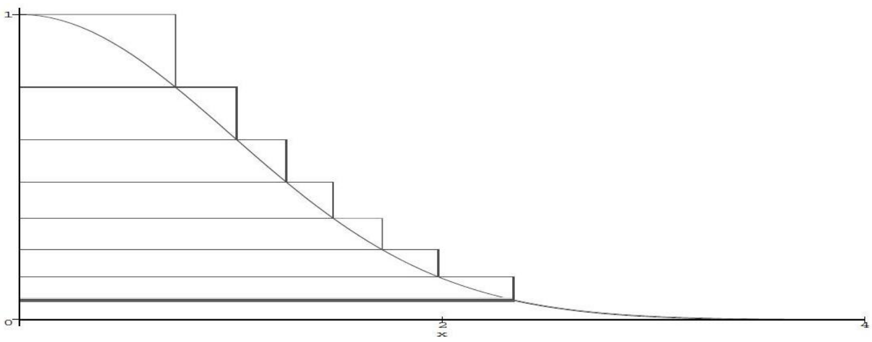


Figure 3.10: The Ziggurat Method: Illustration

Generally, in the practical implementation of this method the covering will be much finer. Usually, the preferred number of covering rectangles will be greatest number that can be stored in 8 bits of memory – 256 – or half of that for symmetrical distributions.

Using this covering, for all rectangles apart from the top one, as well as for the unbounded base, it is really easy to check if a randomly generated point falls within the bounds of the density function. If the x-coordinate of the point lies to the left of the rightmost x-coordinate of the rectangle on top of it, then it should certainly be accepted as it lies within the boundaries of the probability density function. This happens in the vast majority of cases. The only cases when additional, more computationally intensive checks would be required is if the random point ends up in the top rectangle, in some of the lower rectangles but to the right of the rightmost x-coordinate of the rectangle immediately on top, or in some part of the infinite tail of the distribution to the right of the rightmost x-coordinate of the lowest rectangle.

Out of these three scenarios, the generation of a random point from the tail of the distribution is the most serious problem. It has, however, been solved in multiple academic sources, such as for example Marsaglia (1964), who provides a method for generating random points from the tail of the standard normal distribution.

The method outlined in Marsaglia & Tsang (2000) for doing this is as follows. Denote by r the rightmost x-coordinate of the lowest rectangle. Then keep generating random variables $x = \frac{-ln(U_1)}{r}$ and $y = -ln\,(U_2)$, where U_1 and U_2 are two independent uniformly distributed random variables, until the condition $2y > x^2$ is satisfied. Whenever this happens, the random point in the plan will lie within the density function in the tail of the distribution. Then, return $r+x$, which will conform to the probability density function up to a constant.

The procedure described above consists of two distinct parts and is a restatement of the older results available in Marsaglia (1964). The intuition behind this procedure is perhaps best understood within the context of generating a random variate from the tail of a simpler distribution, for instance the exponential. A logical way to accomplish this would be to generate a uniformly distributed random variable along the y-axis and scale it in such a way, that it lies between 0 and the probability density value corresponding to the rightmost coordinate of the lowest covering rectangle r. Then, wherever this point happens to lie on the y-axis, one can easily obtain a random variate from the tail of the desired distribution by simply inverting the probability density function in order to find the x-coordinate corresponding to the randomly generated point on the y-axis. Figure (3.11) illustrates this concept graphically. For the purposes of this illustration, an r value of 3 is chosen and the random point on the y-axis happens to be equal to 0.02. The exponential probability density function is then inverted in order to find the corresponding x-coordinate.

Using this method, it is easy to obtain a closed-form expression for generating random variates from the tail of the exponential distribution: simply scale the uniform random variable U by e^{-r} and set it equal to the exponential density of the x-coordinate: $e^{-x} = e^{-r}U$. Solving for x yields: $x = r - ln\,(U)$. The resulting variable x will be random and will be exponentially distributed.

The same principle can be used for generating a random variate from the tail of a normal distribution. For a family of normal distributions with probability density functions $ce^{-\frac{x^2}{2}}$, where c is a standardizing constant, we can use

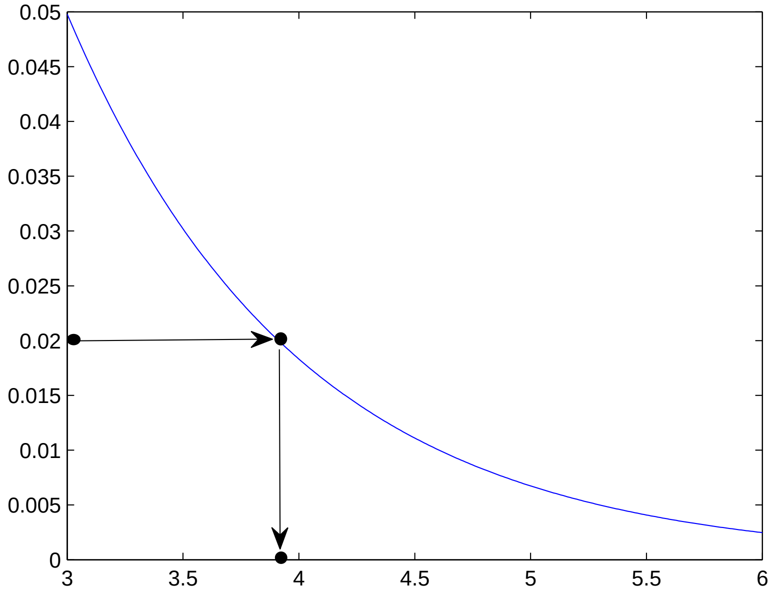


Figure 3.11: Generating a Point from the Tail of the Exponential Distribution

the same method as above: $e^{-\frac{x^2}{2}} = e^{-\frac{r^2}{2}} U_1$. Solving for x gives the required normally distributed random variable, conditioned on x being greater than r:

$$x = \sqrt{r^2 - 2ln\left(U_1\right)}, \tag{3.111}$$

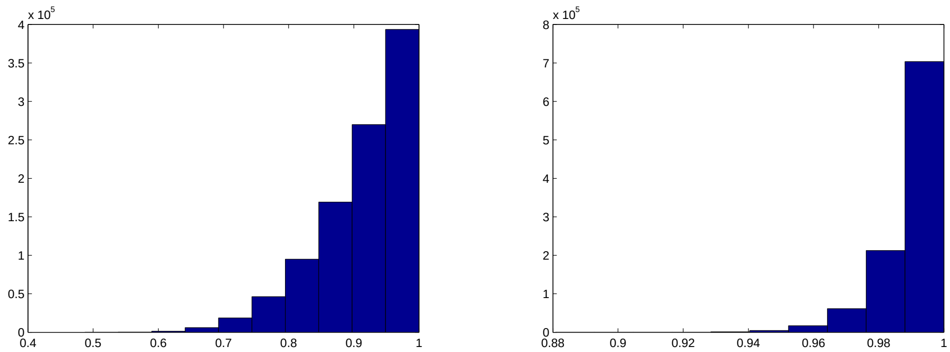
where U_1 is a uniformly distributed random variable. This result is the same as the second equation in formula 1 in Marsaglia (1964).

The first equation in this formula is the second part of the procedure for generating random variates from the tail of the normal distribution – the acceptance-rejection procedure. Unlike some of the other methods for generating normal variates, this is not a test on magnitude to see if a random point in the plain falls within the probability density curve, but is a condition which measures how likely it is to generate a point that will be in the tail of the distribution, given the rightmost x-coordinate r. The condition that needs to be satisfied in order for this test to be accepted is:

$$U_2 < \frac{r}{\sqrt{r^2 - 2ln\left(U_1\right)}}, \tag{3.112}$$

where U_2 is another uniformly distributed random variable, independent from U_1.

What equation (3.112) accomplishes is that it takes the x-values that equation (3.111) generates and by dividing r by them it produces a value in the range between zero and one. This value is then compared to a uniformly distributed random variable. The right-hand side of equation (3.112) also takes account of the distribution of the x-values conditional on the value of r. Whenever, r is large, the greatest probability concentration of x values will be close to it, with the probability of even larger values falling towards zero very quickly. This results in a much steeper distribution with the majority of its probability concentrated around 1, making the condition in equation (3.112) more likely to be true. Alternatively, for smaller values of r, the probability distribution of the right-hand side of equation (3.112) is sloping more gently and has more of its probability further away from one, making the condition less likely to be true. This is graphically illustrated in figure (3.12). The two graphs show the



(a) $r = 3$ (b) $r = 10$

Figure 3.12: Conditional Distribution of r Divided by Normal Tail Variates

distribution of the right-hand side of equation (3.112), conditional on the value of r. The two distributions are for values $r = 3$ and $r = 10$ respectively.

Depending on the value of r, this acceptance-rejection procedure can be highly efficient. Marsaglia (1964) estimates that for $r = 3$ the probability of the event in equation (3.112) is 88%. Usually, the r values for a normal distribution are higher than 3. If 256 covering rectangles are used (255 actual rectangles plus the unbounded base), the rightmost x-coordinate is $r \approx 3.65$, and if the number of covering rectangles is 128, then $r \approx 3.44$. This leads to an even higher efficiency for the acceptance-rejection procedure, which almost always successfully passes the first trial.

In fact, finding appropriate values for the rightmost x-coordinates of all rectangles, not just the lowest one, is an important task in itself. These x-coordinates must be chosen so that the common area of the rectangles is the same as the area of the unbounded base. This is no easy task and even being given a value for the area of the rectangles does not allow finding a closed-form expression for their x-coordinates. Instead a recursive computational trial and error procedure can be used to determine the rightmost x-coordinates of each rectangle. If 255 covering rectangles are used, this method can be summarized as follows: keep trying values for $r = x_{255}$ until the processes

$$v = rf(r) + \int_r^{\infty} f(x)\,dx; \quad v = x_i \left[f(x_{i-1}) - f(x_i) \right],\ i = 254, \ldots, 1,$$

yield a value for the rightmost x-coordinate of the top rectangle x_1, such that $x_1 (f(0) - f(x_1)) = v$. In the equations above, v denotes the common area of the rectangles and the unbounded base. and $f(\cdot)$ denotes a probability density function.

After performing these calculations, Marsaglia and Tsay (2000) report that for the nonstandardized normal density $f(x) = e^{-\frac{x^2}{2}}$ and 255 covering rectangles, the appropriate rightmost x-coordinate is $r \approx 3.65$. This gives a common area $v \approx 0.0049$. Since the total area under the density curve is:

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi},$$

then it turns out that the proportion of time when we will be able to accept a generated random point in the plain on the basis that it lies to the left of the rightmost x-coordinate of the rectangle on top of it is:

$$\frac{\sqrt{2\pi}}{2 \times 256} \div 0.0049 = 99.33\%.$$

This quantifies the efficiency of the ziggurat algorithm since the acceptance-rejection decision is straightforward almost always and more complicated fall back algorithms are necessary for only a tiny fraction of all situations.

Another factor, which contributes to the efficiency of the ziggurat algorithm is the fact that not only is the acceptance-rejection decision very time-efficient most of the time, but also it can be made by generating only a single random number rather than two. This is accomplished by floating a random 32-bit unsigned integer in order to produce a uniformly distributed random variable as a part of the algorithm itself, which allows the random integer to be re-used for two different purposes.

This is accomplished by means of a separate initialization phase in the implementation of the pseudo-random number generators. This initialization function is run only once in the beginning of a simulation, and produces two tables of values. Subsequently, multiple random numbers can be generated very quickly by taking values from these two tables whenever necessary. The tables are obtained in the following way: for each of the rectangles $1 < i < 255$ set a 32-bit unsigned integer $k_i = 2^{32} \left(\frac{x_{i-1}}{x_i} \right)$, and for the second table, set $w_i = \frac{x_i}{2^{32}}$. For the special case when $i = 0$, set k and w values for the unbounded base:

$k_0 \left[\frac{2^{32} r f(r)}{v}\right]$ and $w_0 = \frac{v}{2^{32} f(r)}$. Whenever a number is multiplied by one of the k values it is scaled up, and conversely it is scaled down when multiplied by one of the w values.

In the rare cases when the acceptance-rejection procedure fails, there are only two alternatives to consider: either the generated number falls to the right of the rightmost x-coordinate of the rectangle immediately on top of it, or it is in the tail of the distribution. If it is the former, then an additional test is performed to check if the generated value falls under the distribution density. This requires the generation of one more uniformly distributed random variable. In situations when the latter occurs, then a value from the tail of the distribution is returned. As discussed above, this entails the generation of at least two more independent uniformly distributed random variables and the taking of natural logarithms. By definition, whenever the top rectangle is selected, it falls in the first category of cases and is handled in the same manner.

The implementation of the ziggurat method for generating random variables from a required distribution can be summarized using the following sequence of steps:

1. Generate a random 32-bit unsigned integer j and take its 8 least significant bits. Denote this index by i. This number will determine which covering rectangle is selected. The generation of the random integer is implemented by means of a three-shift shift-register generator. Given a seed, usually provided by the system clock, it performs three sets of bitwise operations on it in order to produce a pseudo-random integer. Firstly, it shifts the number by 13 bits to the left, performs a bitwise exclusive or operation and saves the result. This is repeated twice more, first moving the number 17 bits to the right, then moving it again 5 bits to the left. Random numbers produced in this way will have a period of $2^{32} - 1$.

2. Scale the random integer down, so that it can be compared to the x-coordinate of the rectangle immediately on top of it: $x = jw_i$. Then compare and see if it falls to the left or to the right of it. The table with the k values can be used for this. If $j < k_i$, then the acceptance-rejection procedure is successful and x will be a random variate from the desired distribution. Using the k and w tables and by using the last 8 bits of the random integer, it is first used in order to select a rectangle and then reused to compare the x-coordinates.

3. If the acceptance-rejection procedure fails, i.e. if $j \nless k_i$, a separate function is called to handle these unusual situations. Again, two alternatives are possible:

(a) If $i = 0$, then the unbounded base has been selected and the random number is to the right of the rightmost x-coordinate of the lowest rectangle r. In such a case, simply return a random variate from the tail of the desired distribution using the method outlined above.

(b) If $i \neq 0$, then we are either in some of the rectangles but to the right of the x-coordinate of the rectangle immediately on top, or in the top rectangle. We handle these cases in the same manner by generating another uniformly distributed random variable U. If $[f(x_{i-1}) - f(x_i)]\,U < f(x) - f(x_{i-1})$, then the generated x lies below the density of required distribution and can be returned. If the above condition is not satisfied, this value of x is discarded and we go back to step 1. This sequence of steps is repeated until a value for x is finally returned.

Apart from pseudo-random number generation our implementation of the simulation model includes a number of ancillary utility functions. These are used mainly for early error detection and prevention. They are also useful in warning the user about undesirable program behaviour and potentially inappropriate parameter values. Error checking and prevention of mistakes during the input of parameters by the user is also implemented in order to avoid taking in parameter values of a type different than the required one.

In this section we gave an overview of the implementation of our simulation model and stressed some of its important points, as well as some of the problems that may arise and choices that have been made. The main building blocks of the implementation are the definition of classes for the convenient storage and manipulation of inputs and outputs, a set of procedures for the efficient generation of pseudo-random numbers, and a number of functions implementing the core program functionality. The latter include functions that handle the simulation of the regime switching process, the dividend generating processes, estimation of conditional regime probabilities, as well as the calculation of fundamental values under certainty and uncertainty. The outputs of all these functions are then used during the implementation of the main wealth dynamics simulation. With this section we wrap up the model chapter and move on to the analysis of the simulation results.

Chapter 4 Overview of Results

Contents

In this chapter we provide an overview of some of the outputs from the simulation model discussed in the previous chapter. While the results of our agent-based model of market dynamics are not normative, in the sense that no explicit optimal asset allocation strategy in the context of fixed minimum payout liabilities can be prescribed on the basis of our results, we do attempt to isolate some features of investment behaviour that provide a better chance of survival for large institutional investors during recessionary periods of asset price deflation and negative stock market returns. Consequently, much of our analysis will be focused on times of negative economic climate. Even so, any type of behaviour that is successful during recessionary periods must not severely compromise an institutional investor's performance during normal economic conditions, since the latter predominate in the long run. Hence, non-recessionary periods will not be completely discarded either.

The outline of this chapter will concentrate mostly on the results that follow from the innovations in our model. Namely, the inclusion of a minimum consumption constraint, which creates the possibility of bankruptcy, the inclusion of a simplified two-regime business cycle, as well as the investor's ability to learn about the regimes of an industry and to base their investment behaviour on their ability to estimate fundamental values,

The main ambition behind the model presented above is to explore the results of enriching a particular class of evolutionary finance models by adding more realistic constraints. Our desire with this exploration is to use it in order to understand how the additional parameters and constraints interplay with the basic financial market dynamics of previous evolutionary finance models. Namely, the model can be used to examine the following questions:

- The connection between strategy aggressiveness and investor performance. Strategy aggressiveness here refers to two aspects of investor behaviour. Firstly, it may refer to an investor's propensity to aggressively move in and out of positions by frequently and significantly adjusting one's asset holdings. Secondly, strategy aggressiveness may also refer to exposure to market risk, measured by the average relative proportion of risky assets in an investor's portfolio. The effect of both of these interpretations on investment performance can be analyzed.

- The link between the number of available assets and the volatility of assets in the market. It can be investigated whether an increased number of risky investment opportunities has an impact on the frequency of switching between them, thus contributing to market volatility.

- The conditional probabilities of bankruptcies. There is scope for investigating how likely it is for a certain investor type to survive during difficult economic conditions when at least one bankruptcy has occurred. Additionally, the conditional probability of bankruptcies exclusively for the recessionary regime of each industry can be considered.

- The ranking of investment strategies and the probabilities of the different investment styles to outperform the others. An important question to consider is how the dominance of a certain investment style can have important implications for market efficiency or the formation of asset price bubbles.

- The first passage time until the minimum consumption constraint is activated. This is of particular importance if conditioned on the industry being in the recessionary regime since it will give a rough estimate of the amount of time investors have until they start paying the minimal guarantee after the industry has moved in recession. This estimate may help in the investors' risk management activities by allowing them to make contingency arrangements before the expected large increase in proportional payouts.

- The expected time to bankruptcy. This metric will be a measure of how long it takes on average for any of the investors to reach insolvency. This scenario is important because a bankruptcy of one of the investors always destabilizes the market at least temporarily, and brings both additional risks and opportunities for the survivors.

- The impact of the minimum consumption constraint on the market dynamics. The sensitivity of the model's results to changes in the additional constraint can be investigated, with regard to not just the performance of each individual investor type but the financial market as a whole.

- The impact of the riskiness in the business cycle. Another interesting question that can be investigated is how different features of the business cycle affect investor behaviour and the market dynamics. For instance, an important measure of the inherent riskiness of the business cycle is the severity of recessions – i.e. the difference between income levels during normal and recessionary periods. The more pronounced this gap is, the greater the risk and the greater the potential impact that the additional consumption constraint may have.

We now proceed to provide an illustration of the different components of our model during a single simulation run in order to give a visual perspective of how each of them fits together to create the market dynamics. This is followed by a discussion concerning the choice of appropriate parameters and some difficulties that might be encountered in the process. These turn out to be of crucial importance in addressing most of the questions posed above. We show some deficiencies of the model that make it quite difficult to provide aggregate quantitative data about the model's behaviour or to draw conclusions about sensitivities to certain parameters. We wrap up this section by providing ideas of useful benchmark strategies that are incorporated into the model for the purposes of benchmarking it against popular investment strategies used financial research.

4.1 Illustration of a Sample Run

In this section we present a sample run of all components of our simulation model for three of the most common scenarios for the evolution of the investors' wealth endowments. The purpose of this section is to provide a graphical insight into the workings of the model. In this way we can begin to form a theory of optimal asset allocation under minimum consumption constraints. Generalization of the results presented in this section, as well as the validation of any theory, can be carried out by means of generating many sample runs while perturbing some of the model parameters and examining the statistical properties of the outputs.

The parameters for the sample run illustrations are presented in table (4.1) below. In table (4.1), dt refers to the size of the time step, $nbsim$ denotes the number of simulation runs performed, T is the investment horizon measured in years, DGP and $MRSR$ signify that the mean-reverting square root process was used in order to generate the dividend realizations, $nAssets$ denotes the number of risky assets in the economy, $\underline{\mu_k}$ are the mean-reverting dividend values in the recessionary regime for each of the risky assets, D_0^k specify the starting dividend values for both risky assets, r is the interest rate and c is the consumption proportion, $nInv$ denotes the number of different investor types in the simulation, each with an initial wealth endowment V_0^i and initial asset allocations to risky asset k denoted by $\lambda_{k,0}^i$. The minimum consumption constraint in absolute terms is denoted by m. For the time being we provide no discussion about how these parameters were picked, or about various problems that might arise in this respect. No explicit scenarios for the regime switches were used in these simulation runs.

4.1.1 Favourable Economic Conditions

We begin by considering a case when economic conditions remain favourable for the duration of the simulation and none of the investors faces insolvency.

Firstly, in figure (4.1) we provide a sample realization of the dividend processes for the two risky assets, together with the evolution of the regime switching process.

During this realization, dividends remain mostly in the normal regime and only occasionally dip into short-lived recessions. It is not until the end of the investment horizon that a more pronounced recession begins to take place. This will no doubt affect investor performance but it is not enough to induce any bankruptcies within the predetermined investment horizon.

Based on these dividend observations investors will try to estimate the conditional probabilities of being in each state of an industry. The output of this process is illustrated in figure (4.2). The Bayesian updating of regime probabil-

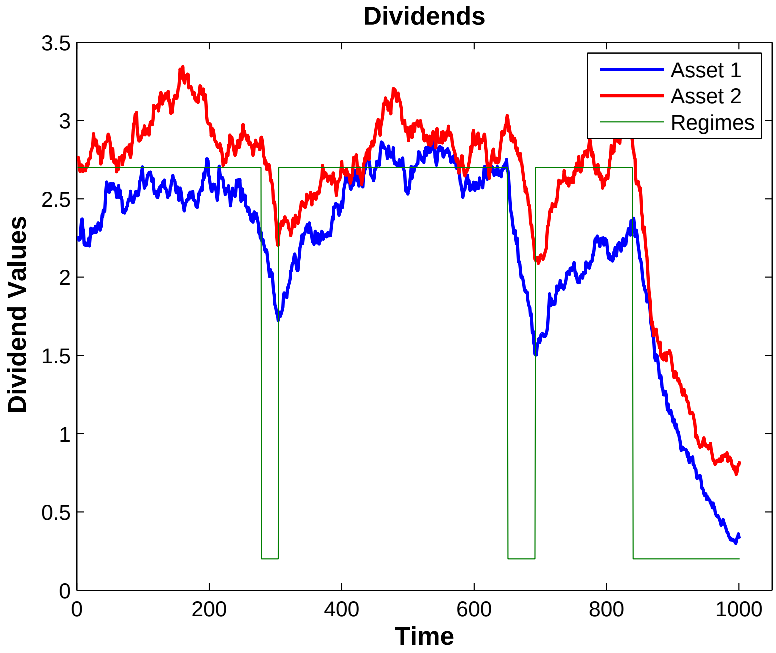


Figure 4.1: A Dividend Realization: Mean-Reverting Square Root Process; Normal Conditions

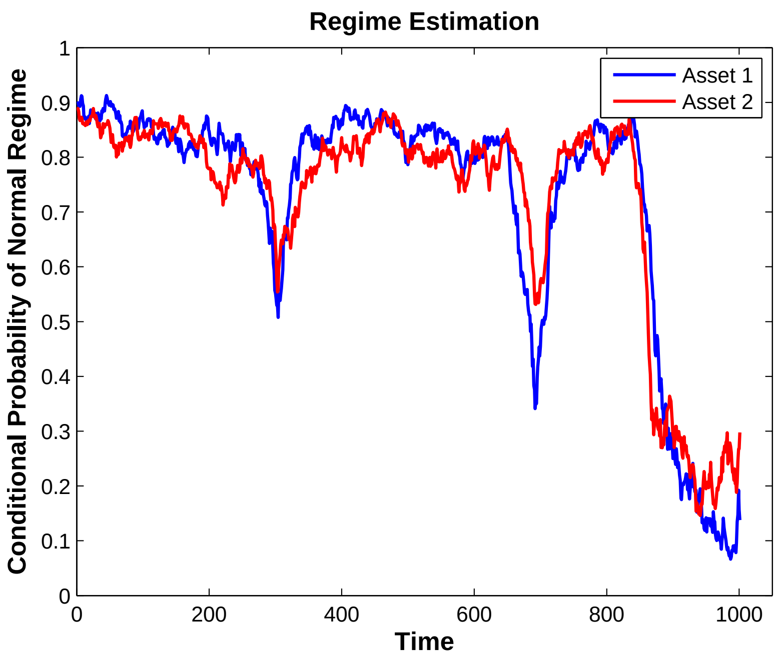


Figure 4.2: Estimation of Regime Probabilities; Normal Conditions

Table 4.1: Simulation parameters for the sample runs.

Parameter	Value
dt	0.01
$nbsim$	30
T	10
DGP	$MRSR$
$nAssets$	2
μ_1	0.2
μ_2	0.7
D_0^1	1.25
D_0^2	1.75
r	0.1
c	0.33
$nInv$	4
V_0^1	3
V_0^2	3
V_0^3	3
V_0^4	3
$\lambda_{1,0}^1$	0.43
$\lambda_{2,0}^1$	0.46
$\lambda_{1,0}^2$	0.43
$\lambda_{2,0}^2$	0.46
$\lambda_{1,0}^3$	0.36
$\lambda_{2,0}^3$	0.38
$\lambda_{1,0}^4$	0.36
$\lambda_{2,0}^4$	0.38
m	0.55

ities results in accurate and responsive estimation of the true regime switching process.

Having estimated the conditional probabilities of both regimes, the investors are in a position to calculate their perceived fundamental values for the two risky assets in the economy. This is illustrated in figure (4.3).

Investors who trade on the basis of deviations from fundamental values can then make their investment decisions and implement their asset allocation policy. The asset allocation process for one of the risky assets is demonstrated in figure (4.4).

The fairly large volatility in the investment actions of the trend follower during the initial stages stems from the fact that he lacks the necessary historical price information to implement a genuine trend following policy during the initial predetermined lookback window.

Once enough price history becomes observable and trend following behaviour begins to emerge, energetic portfolio rebalancing takes place. The latter impacts asset prices as well and contributes to some of the short-term volatility

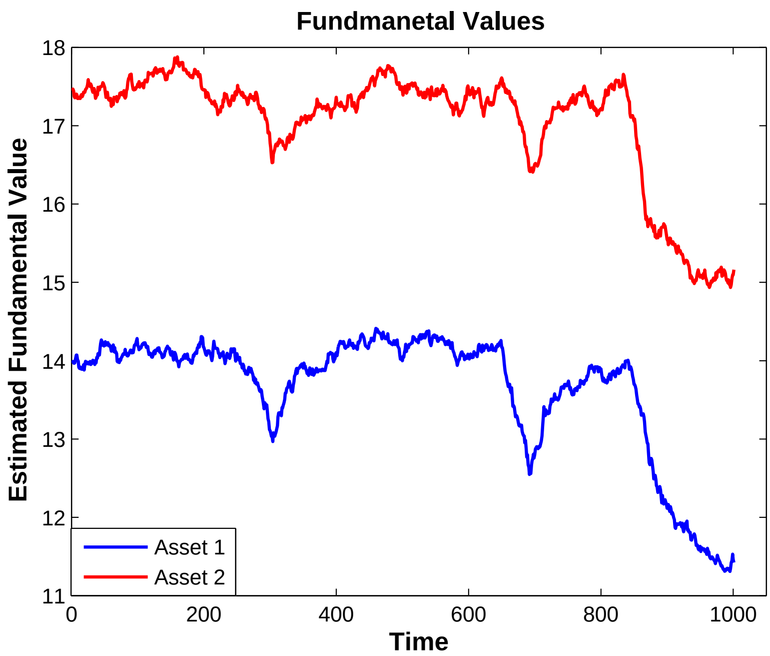


Figure 4.3: Estimation of Fundamental Values; Normal Conditions

in asset prices. The latter, though, is much less pronounced, since the active trading behaviour exhibited by the trend follower is somewhat offset by the relatively more stable asset allocation policies of the other three investor types.

The investors' actions in the marketplace contribute to the asset price formation through the endogenous mechanism described in the previous chapter. The resulting realization of the prices of the two risky assets is shown in figure (4.5).

Once prices are also calculated, the investors' asset allocation policies can be translated to relative asset holdings by each agent as opposed to the amount of wealth allocated to a particular risky asset. The investors' holdings of one of the risky assets measured in terms of the net number of shares owned are shown in figure (4.6).

Despite the relatively stable investment policy of the trend follower towards the middle of the simulation run, the eventual decrease in this agent's wealth endowment leads to decreased market power, measured by an investor's ability to have a large impact on the asset price formation. This results in decreased asset holdings even though the propensity to invest in risky assets, as measured by the investment proportions illustrated in figure (4.4), remains largely unchanged, save for the short-term fluctuations. This interesting phenomenon is caused by the endogenous structure of the asset price formation in our model, which ensures a transfer of wealth from less successful investment styles to more successful ones.

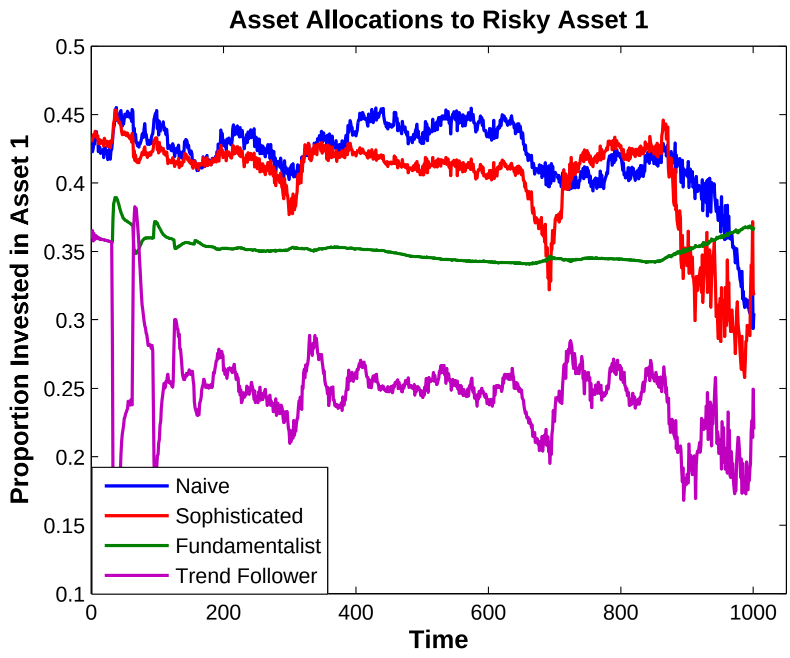


Figure 4.4: Asset Allocation to Risky Asset 1; Normal Conditions

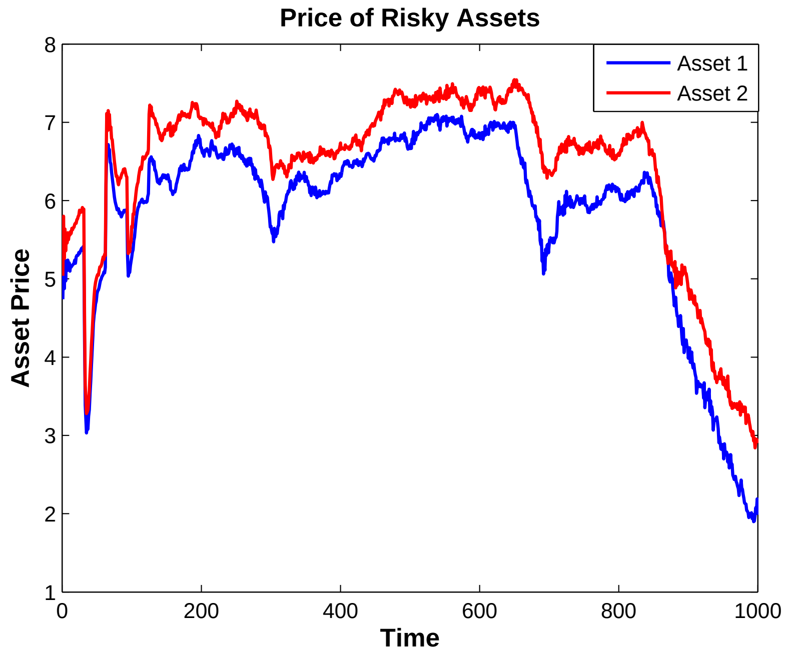


Figure 4.5: Asset Price Dynamics; Normal Conditions

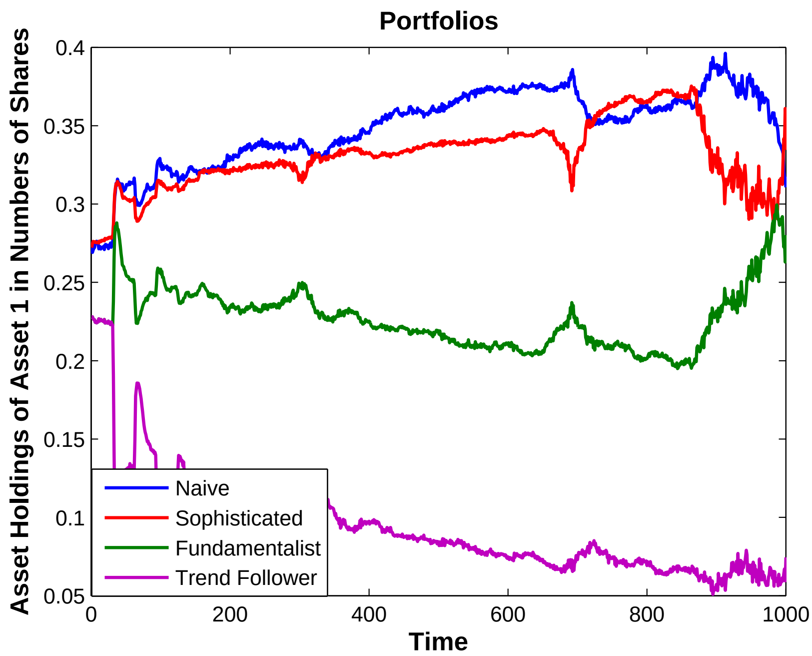


Figure 4.6: Asset Holdings of Risky Asset 1; Normal Conditions

The loss of wealth by the trend follower means that the opinions of the other investment types carry more weight and their actions speak louder. Thus, even though the asset allocation policy of the trend follower remains either stable or slightly decreasing, increasing asset prices during this period of the simulation mean that the other agents are amassing more of the wealth of the market under their control at the expense of the trend follower by means of the acquisition of more shares of the risky assets. That is why it is usually more helpful to analyze asset holdings rather than investment proportions, since the former clearly show which investment style is increasing its holdings of risky assets and which agent is selling risky assets.

This approach to modeling market interactions of investment strategies allows the market itself to determine the optimality of a certain type of behaviour by putting more resources under the control of successful investors. In other words, optimality is measured by an investor's bottom line rather than the maximization of an arbitrarily chosen utility function. While a utility function might depend on a lot of assumptions about economic behaviour, such as risk aversion for example, letting the market itself decide what the best course of action is implies only the assumption that investors prefer more to less.

The process of wealth transfer discussed above is made even more pronounced with the inclusion of the minimum consumption constraint. When the latter is active, the increase in wealth under the management of successful investment styles is much faster, as is the concomitant loss of wealth by

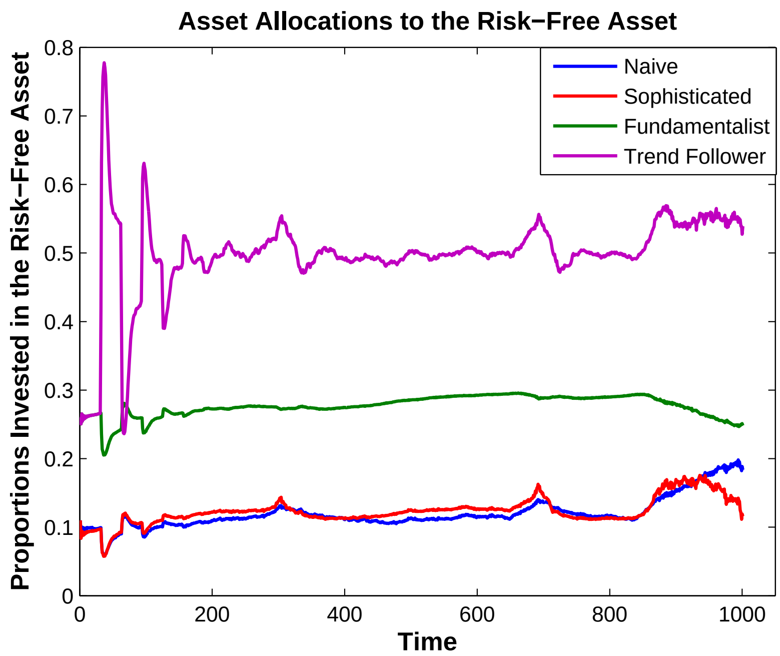


Figure 4.7: Asset Allocation to the Risk-Free Asset; Normal Conditions

poorly performing investment styles. In the extreme case of bankruptcy, all the resources of the insolvent agents are appropriated by the survivors.

The relative cautiousness of the trend following investment style and its aversion to exposure to market risk are confirmed by the fairly high allocation to the risk-free asset in the portfolio of this asset management approach. This is illustrated in figure (4.7). Apart from the unstable period of time, during which the switch between the fundamental and trend following strategies occurs, most of the time the trend follower allocated around 50% of all available resources to the risk-free asset during this particular simulation run.

Since during long periods when the industry is functioning normally there shouldn't be any need for the majority of the investors to quickly liquidate parts of their portfolios under conditions of a "fire sale", there should be no expectation that the majority of the agents will face enough sustained pressure to force them to consume at the level of the minimum consumption constraint m. This expectation is confirmed in figure (4.8), where the minimum consumption constraint is activated only for the trend follower towards the end of the simulation run, owing to the prolonged poor investment performance of this investment style, which decreased the trend follower's wealth endowment to a point, where consuming at rate c was no longer sufficient in order to meet the minimum consumption constraint.

The fact that most of the time consumption is proportional to the size of each agent's wealth endowment leads to the conclusion that the general shape

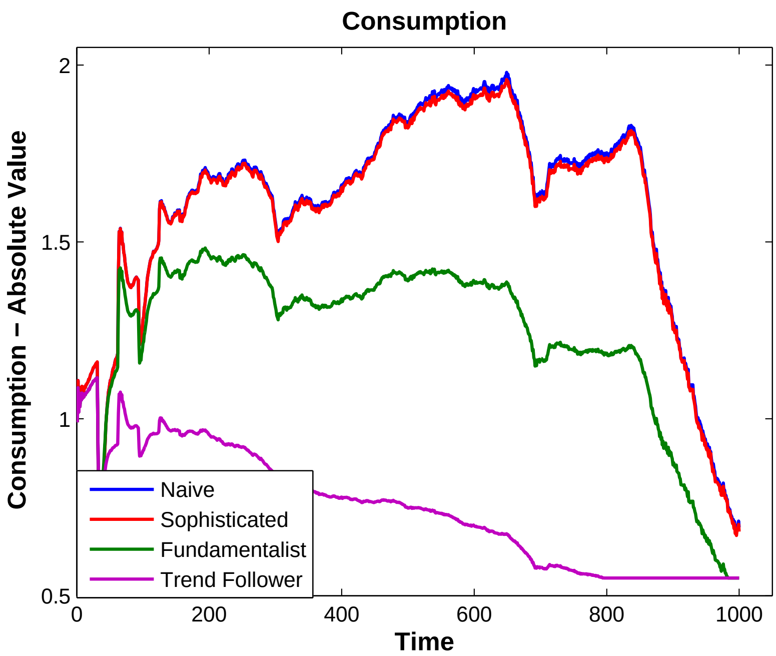


Figure 4.8: Consumption in Absolute Terms; Normal Conditions

of the evolution of the investors' wealth endowments should follow a trajectory similar to that of figure (4.8). This is confirmed in figure (4.9), which illustrates the evolution of the investors' wealth endowments.

While normal economic conditions tend to dominate most of the time, there are situations when prolonged recessionary periods may set in as well. This makes a significant impact on the results achieved by each investment style and has important implications on strategy optimality during severe recessions.

In the following two subsections we present two cases of prolonged recessions. These have very different outcomes in terms of the survivability of the different investment styles, as well as different implications for the economy as a whole. Both rational and irrational behaviour can lead to favourable investment results but at the expense of two largely different scenarios for the financial market. In the following subsection we present a case of investor irrationality leading to a serious economic collapse and complete market failure.

4.1.2 Recession

4.1.2.1 Flight to Safety

In this section we present the first frequently encountered scenario that may occur when investors are faced with a prolonged and severe recession. Figure (4.10) shows the dividend realizations in this case and it is immediately obvious that unlike the previous section, where investors only had to deal with the

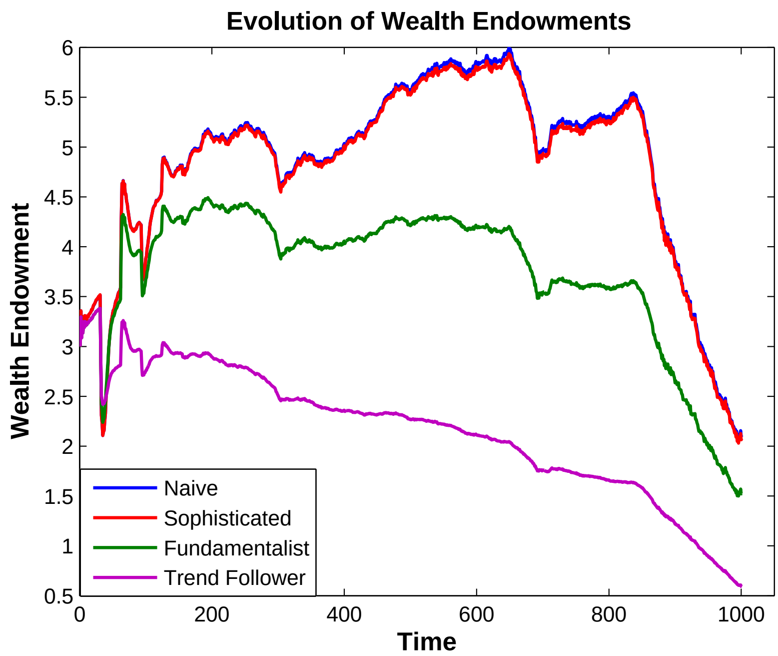


Figure 4.9: Evolution of Wealth Endowments; Normal Conditions

consequences of two mild, short-lived recessions, here they must cope with a sustained slowdown of economic activity.

Once again, investors pick the regime shift quite accurately but they tend to be less certain in their probability estimation, while dividends have spent a prolonged period of time in the recessionary regime. Figure (4.11) clearly illustrates the large oscillations in estimated conditional probabilities whenever dividends make even a slight positive move while in a recession. These small, short-term fluctuations in the dividend process appear much more meaningful to investors in this new context of chronically low dividend income.

The sharp decline in dividends and the relatively higher uncertainty about the prevailing regime in the industry naturally carry over to the estimation of fundamental values, which preserve the shape of the dividend realization, as well as the greater uncertainty during the recessionary regime, although not to such a great extent as the conditional probability estimation. The calculated fundamental values for this simulation run are illustrated in figure (4.12).

As before, the fundamental values and the conditional regime probabilities are then used by the investor types that need them in order to implement their asset allocation policies. These are illustrated in figure (4.13). There are a couple of important distinctions in the the agents' investment behaviour during this scenario as opposed to figure (4.4).

Firstly, in keeping with the discussion in the previous paragraphs, after the sharp decline in dividend income, investors begin to engage in trading activities

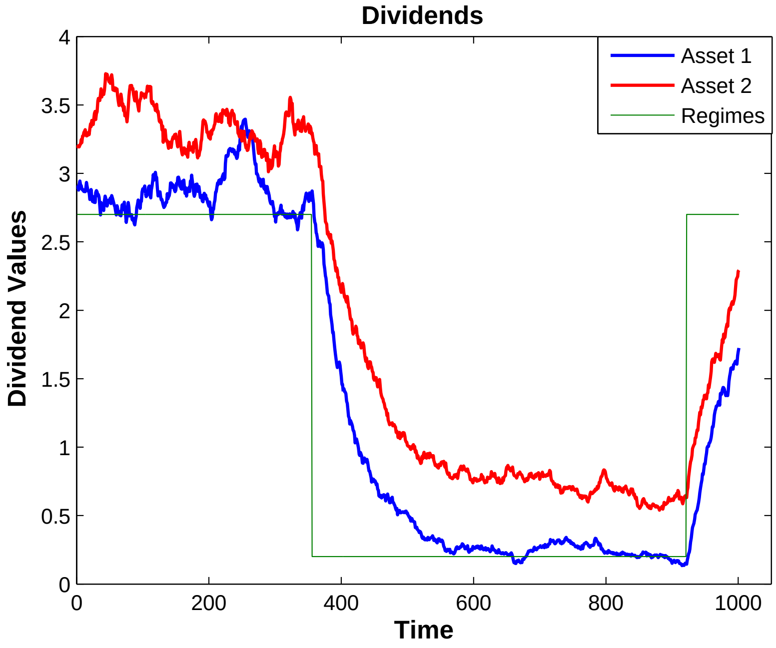


Figure 4.10: A Dividend Realization: Mean-Reverting Square Root Process; Recessionary Scenario 1

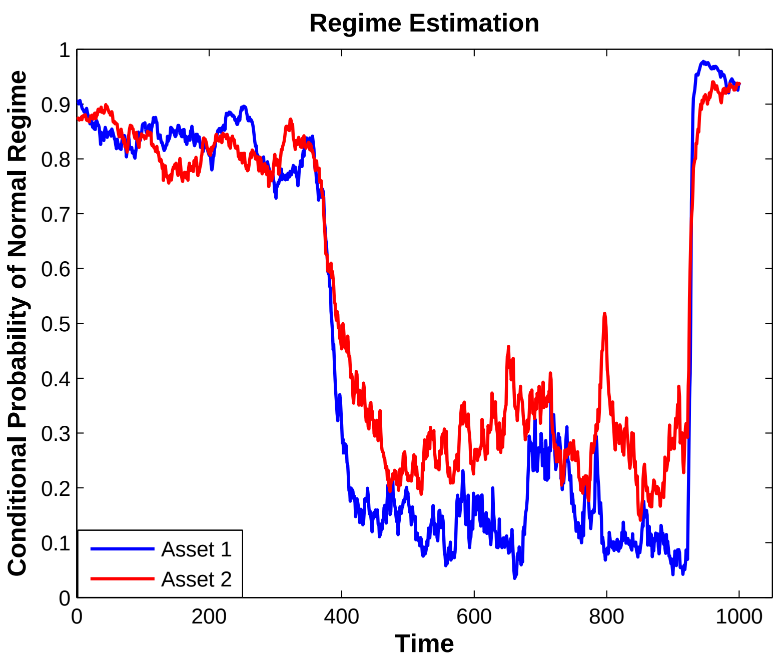


Figure 4.11: Estimation of Regime Probabilities; Recessionary Scenario 1

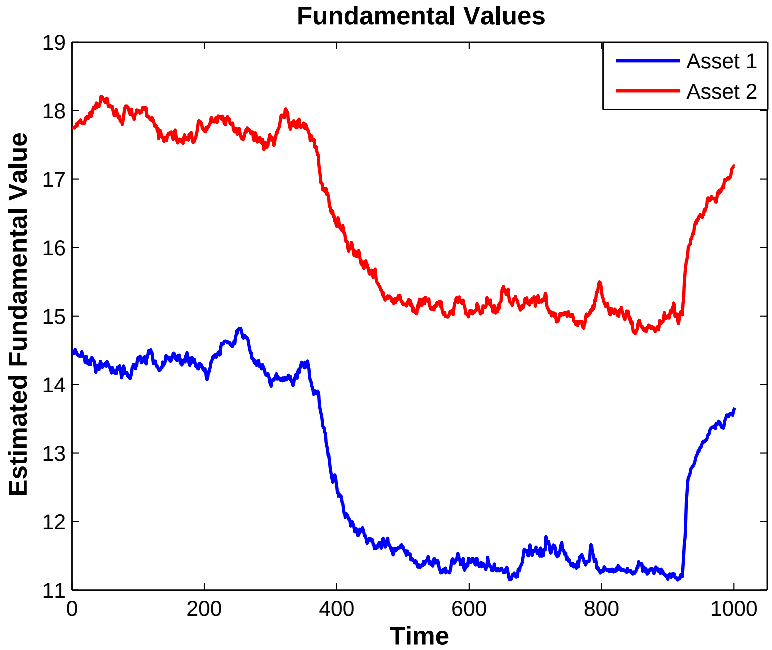


Figure 4.12: Estimation of Fundamental Values; Recessionary Scenario 1

much more vigorously and at a greater scale. This is illustrated by the large volatility in trading strategies after time period 400, when the recessionary regime becomes active. This increase in uncertainty and anxiety demonstrated by agents is entirely in line with many empirical artifacts of modern financial markets, such as for example the negative correlation between option implied volatilities, measured by the VIX, and stock market movements. The outputs of our model, are therefore capable of closely replicating the reality of financial markets.

Another important feature is the change in investor behaviour when all agents populating the model reach the stage where they have to consume at the minimum consumption constraint m. As will become evident in the following paragraphs, this occurs around time period 600. The investment activities of some agents change around this time as well. While the trend following style has been mostly moving out of the risky assets and into the risk-free asset, this decline in risky asset allocations is not so pronounced before time period 600 and is quite volatile. Once all investors are consuming at level m their wealth endowments are drained at a higher rate than before and they must scramble to sell part of their portfolios under conditions of a "fire sale" to meet these greater demands. This results in a steep asset price devaluation, which in turn causes the trend follower to stay out of the market for risky assets. The trend following investment strategy begins to decrease its allocations to risky assets at a greater rate and with far less volatility. This process continues to accelerate

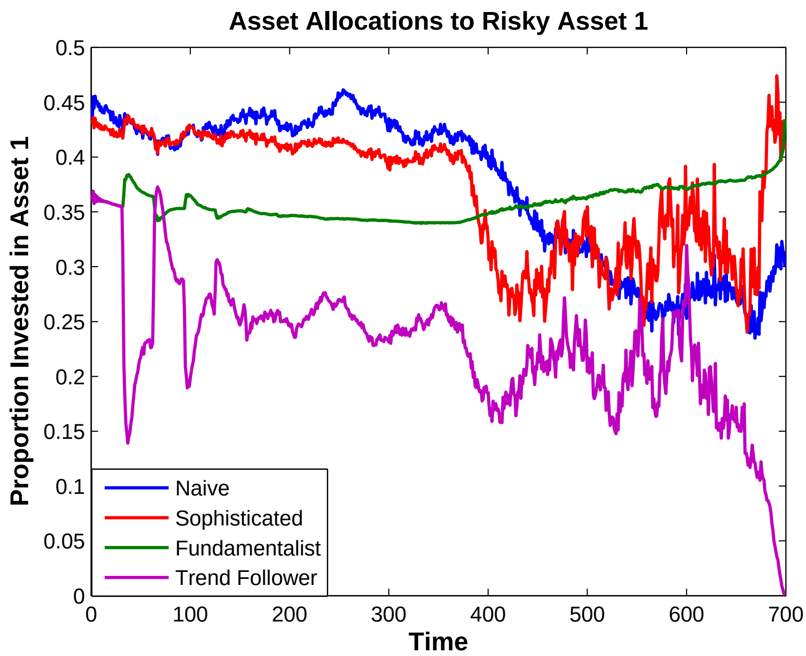


Figure 4.13: Asset Allocation to Risky Asset 1; Recessionary Scenario 1

until the very end of the simulation run.

The complete opposite behaviour is observed on the part of the fundamentalist. This investment style, which remains quite stable for the first part of the simulation run, begins to smoothly increase its allocations to the risky assets as the recession takes full effect around time period 400 and asset prices begin to decrease. This is interpreted as undervaluation by the fundamental trading style and it starts taking advantage of it. Similarly to the trend following style, the fundamentalist's trading decisions begin to increasingly favour one side of the market, but conversely to the trend follower, the fundamentalist seeks to rapidly increase the share of available resources allocated to the risky assets. This is especially pronounced around time period 600, when the switch to consumption at the minimum level m causes asset price deflation to occur at a faster rate, thus further encouraging the fundamentalist to invest more in risky assets.

The investment strategies of the two dividend yield investors remains largely unchanged. Both strategies become more volatile during the recessionary period, the sophisticated dividend yield style much more so, owing to the greater uncertainty about the prevailing regime of the industry. Apart from that, however, no significant change in policy is observed when the recessionary period becomes active and when all investors start consuming at level m. If anything their allocations to the risky assets are slightly increased since dividends have reached their recessionary mean-reverting level and are relatively stable at it,

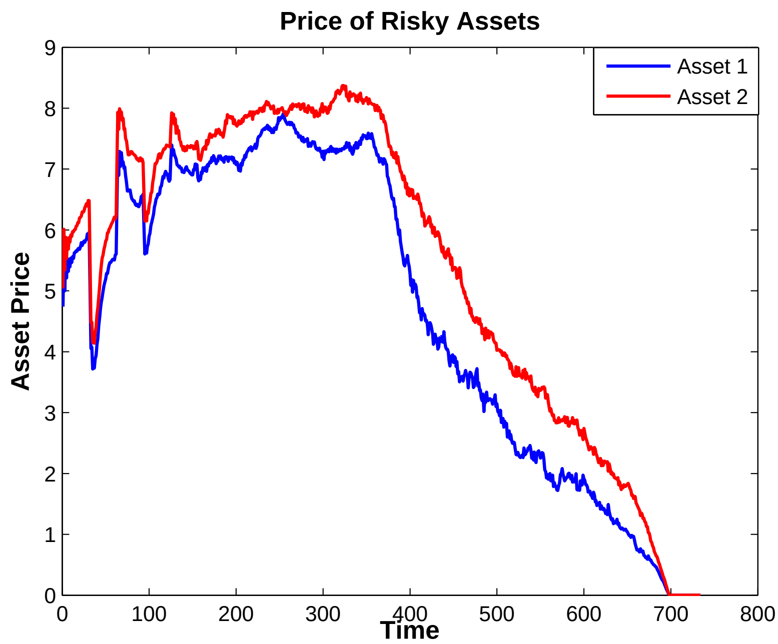


Figure 4.14: Asset Price Dynamics; Recessionary Scenario 1

while asset price deflation continues as all investors consume at the higher percentage m.

This severe drop in asset prices is illustrated in figure (4.14). The devaluation in asset prices was caused by two main components. The first is the obvious large decrease in dividend income during the recessionary period. The decreased size of investors' wealth endowments is translated by the endogenous price setting mechanism to a prolonged asset price devaluation. This cause accounts for the majority of the observed decrease in asset prices.

Perhaps more importantly, however, is the second component, which becomes active around time period 600 – the time when all investors are forced to consume at level m. This causes asset prices to decrease at an even higher rate. The latter eventually causes a deflationary spiral and this process continues accelerating until the end of the simulation run. The implications of the minimum consumption constraint are particularly well illustrated by the second risky asset during the final stages of the simulation, when its price drops down towards zero almost vertically with very little volatility.

Even though in the discussion above we stressed the fact that if the set of parameters is appropriately chosen, at least a single agent is always guaranteed to survive, usually we will not continue a simulation run until the end of the predetermined investment horizon in the event that there is a single survivor. The first reason for this is that after a single investment styles has gathered all resources in the market and consumption is guaranteed to never surpass

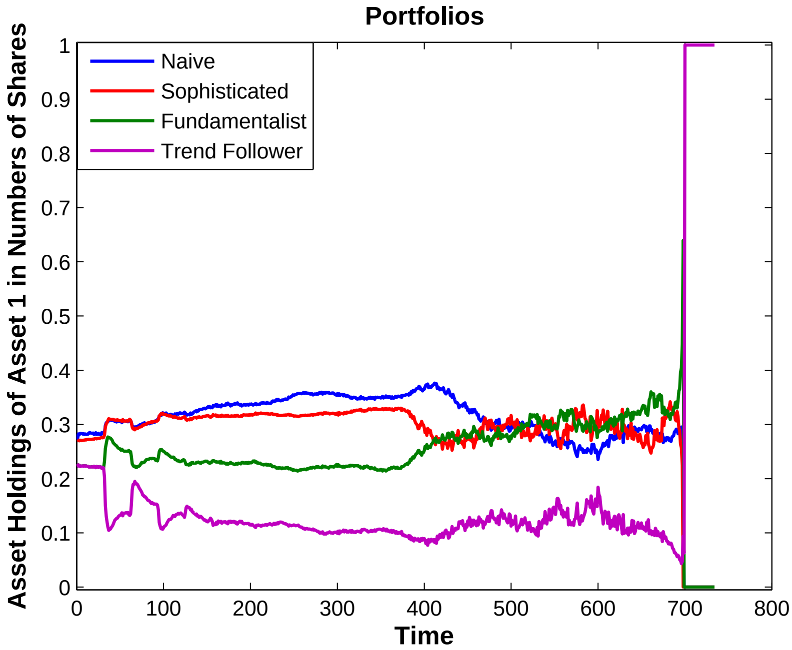


Figure 4.15: Asset Holdings of Risky Asset 1; Recessionary Scenario 1

dividend and interest income, the only situation that can occur in the future is for the wealth of the single survivor to continue growing without bounds. This is hardly interesting from an analytical point of view.

The second reason is purely technical and occurs during simulation runs when only the trend follower remains solvent. Because of the nature of this investment strategy, prices will eventually reach zero during prolonged recessionary periods, such as the one illustrated in this subsection. When this happens, it will be no longer possible to implement the trend following strategy since this would imply division by zero. Therefore, simulation runs with a single survivor will be discontinued once the asset prices reach zero, which would correspond to a complete market breakdown, or when the wealth of the single survivor grows very large and becomes a multiple of this agent's initial wealth endowment.

The mechanism, through which the transfer of wealth from poorly performing agents to the survivors takes place can be further explained by considering the number of shares of each risky asset held by each of the investor types throughout the simulation. The investor portfolios are shown in figure (4.15).

The dynamics of the portfolios shown in figure (4.15) reveals an important feature of the wealth transfer mechanism – it is not a gradual process. The transfer of resources towards the trend following strategy happens instantaneously following the bankruptcy of the three other investor types, even though this trading style slowly begins to amass a greater number of shares when the recession becomes active around time period 400. However, when all investors begin consuming at the minimum level m, the relative asset holdings of the trend follower actually decrease slowly. This is caused by the fact that by this time, the trend following strategy has almost completely shifted its asset allocation to the risk-free asset. Therefore, even though asset prices are declining, it is not enough to allow accumulation of risky assets with such a low percentage of wealth invested in them. This complements the fact that throughout the simulation the wealth endowment of the trend follower stays below the others, so this investment style has less market power in the asset price formation.

In fact, it is exactly this feature of the trend follower's behaviour that allows this strategy to survive and accumulate all the wealth in the market. Had the trend follower started accumulating larger numbers of shares gradually, he would have been increasingly exposed to market risk and would have ended up just like the other three investor types. Instead, the mechanism of wealth accumulation during periods of severe recessions is to move to safe investments and bide one's time until their competitors have disappeared from the market and then acquire their resources at deeply discounted values. In essence, a "phase transition" occurs, which completely changes the level of success in the market.

This scenario is not unlike the tumultuous events of late 2007/2008, when the global credit crisis took effect in a very short amount of time. In the context of our model, the collapse of the real estate market had the same effect as a sharp decline in dividend income. At the same time, the liabilities that the banks had on their balance sheets did not decrease, so some institutional investors entered a phase where their liabilities represented a larger relative share of their income. Faced with the inability to raise new capital, many of them had to file for bankruptcy and were acquired by competitors, who had not enjoyed their level of success during the previous period of favourable economic conditions.

It must be stressed, however, that even though trend following behaviour was successful in this case, it is irrational, in the sense that it allocates resources contrary to deviations from intrinsic value. Furthermore, such behaviour could very well leave an investor in a bad position, should the industry recover, since trend following emphasizes safe investments during any recessionary period. Figure (4.16) illustrates this.

The lower odds of success for the trend follower during favourable economic conditions is confirmed by figures (4.17) and (4.18), showing the evolution of consumption and the wealth trajectories. It is interesting to note that the size of the wealth endowment of the trend follower remains less than that of other investor types throughout the entire simulation run, save for the phase

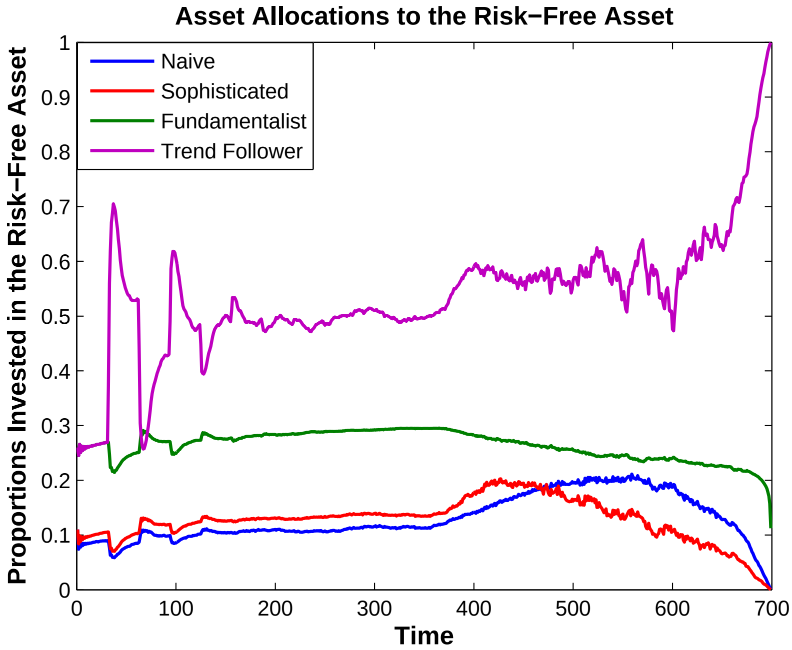


Figure 4.16: Asset Allocation to the Risk-Free Asset; Recessionary Scenario 1

transition in the end, when the three other agents face bankruptcy.

In figure (4.18), the evolution of the wealth dynamics is terminated once prices have reached zero and stayed there for long enough not to allow the calculation of investment proportions according to the trend following style. This, however, is not a concern since by this time, the trend following strategy has all of its resources invested in the risk-free asset and with asset prices being zero, its wealth endowment will simply continue increasing deterministically according the prevailing risk-free interest rate.

In essence, while trend following behaviour proves to be very efficient in safeguarding an investor's funds during periods of severe recession it should not be considered optimal under all circumstances. In fact, the following sections will clearly demonstrate that during periods of favourable economic conditions, the trend following asset management style faces the highest conditional probability of default. It also runs a serious risk of underperforming the other investor types during such market conditions, since by definition it starts participating in trends relatively late, rather than looking for hidden value and positioning itself for expected future favourable movements in asset values.

Trend following achieves personal security at the expense of great damage for the economy as a whole. This trading style has traditionally been associated with herding behaviour in financial markets, either exhibiting itself in the form of asset price bubbles or in the form of major stock market crashes caused by widespread panic. The dominance of this type of investment behaviour is also

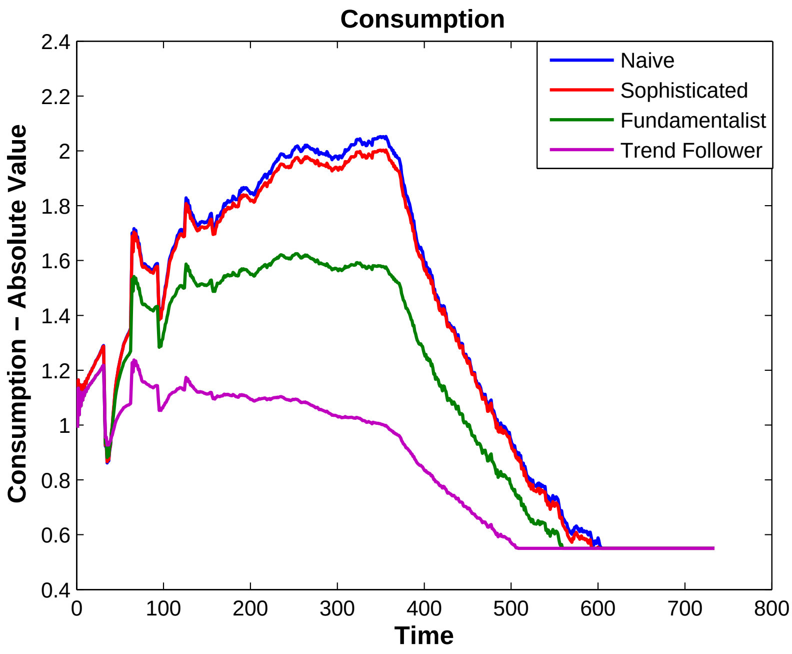


Figure 4.17: Consumption in Absolute Terms; Recessionary Scenario 1

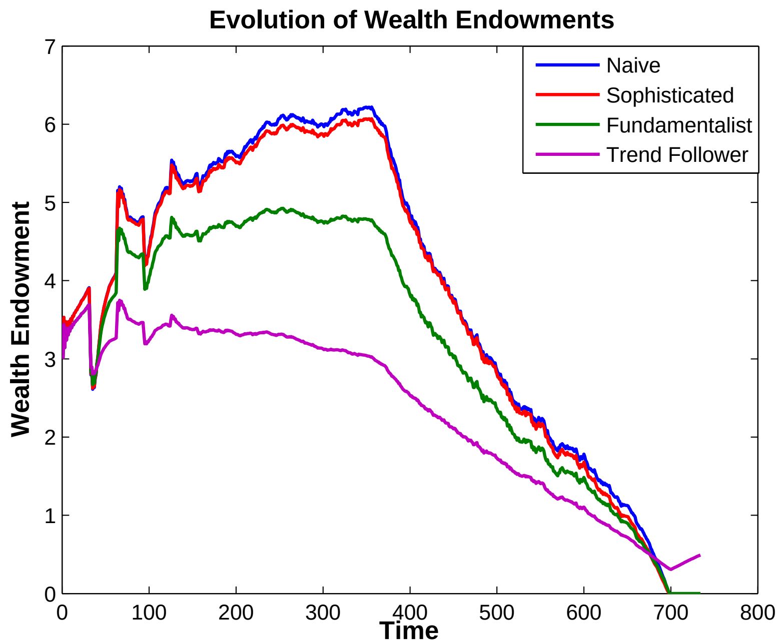


Figure 4.18: Evolution of Wealth Endowments; Recessionary Scenario 1

responsible for the lack of market efficiency, since asset prices will continue to be pushed further away from fundamental values if the main market power lies with the trend followers. The case when this type of behaviour predominates during conditions of already severe recessions is of particular concern, since it underlines the fragility of modern financial systems. The phase transition that occurred within the context of our model could just as well be a sudden bank run or a stock market crash in the real economy.

In addition to this, it is not always guaranteed that the trend following investment style will outperform the others even during periods of severe recession. As already noted, the trend following strategy will move almost completely out of the risky assets during periods of steep asset price devaluation and will therefore have very little market exposure. The transfer of wealth to this investment style happens only after the other agents have faced bankruptcy. Consequently, if some of the other investment styles manages to survive until the end of the recessionary phase, it will be positioned much more favorably for the upcoming economic recovery. This will be the main point illustrated in the following subsection.

4.1.2.2 The Dominance of Value Investing

In this subsection we present the second of the two frequently occurring scenarios that might unfold whenever investors are faced with a prolonged recession. The difference in this case is that the industry ultimately manages to recover before the naive and sophisticated dividend yield investors, as well as the fundamentalist, face bankruptcy.

Since the trend following strategy does not smoothly increase its exposure to risky assets during the period of time when all investors start consuming at the minimum consumption level m, the phase transition described in the previous section does not occur. Instead, the economy manages to bounce back just in time to give investors with high risky asset allocations better chances of survival. This is, however, detrimental to the success of the trend following investment style, which is not positioned favorably in order to take advantage of the economic recovery.

The outcome of these recessionary scenarios is significantly different to that shown in the previous subsection. Investor types which increase their allocations the most during the recessionary asset price deflation phase stand to gain the most from the economic recovery and ultimately remain as the sole survivor in the market.

The evolution of economic conditions is illustrated in figures (4.19) and (4.20). This pattern of scenarios is similar to the previous one in the sense

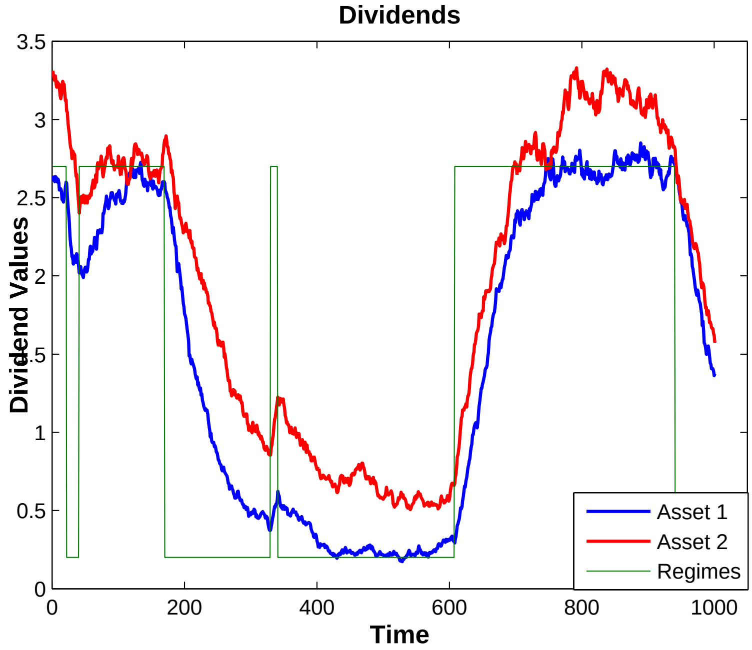


Figure 4.19: A dividend realization under the conditions of a prolonged recession and subsequent recovery. Simulates equation (1.15) with mean-reverting parameters for the recessionary and favourable regimes $\alpha = 1.5$, $\alpha = 1.2$ respectively, and a volatility parameter $\beta = 0.2$.

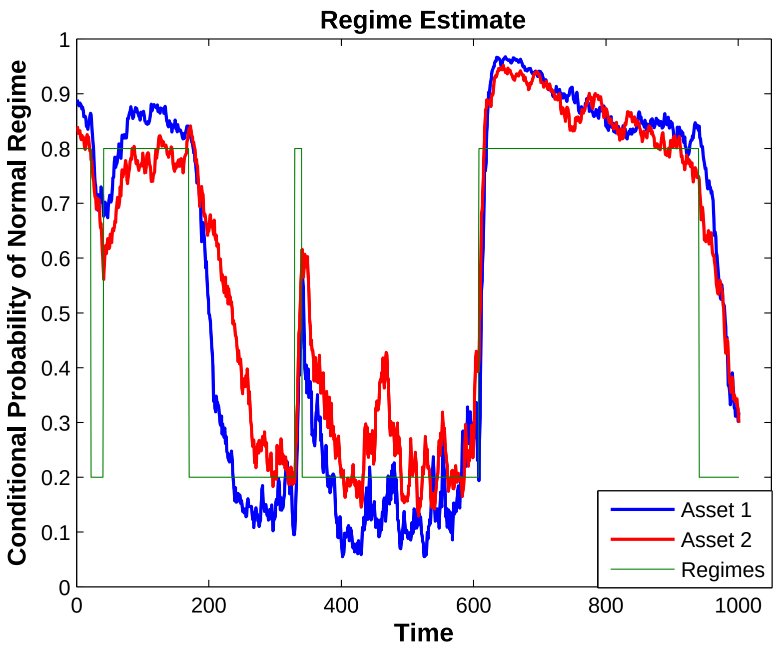


Figure 4.20: Estimation of Regime Probabilities; Recessionary Scenario 2

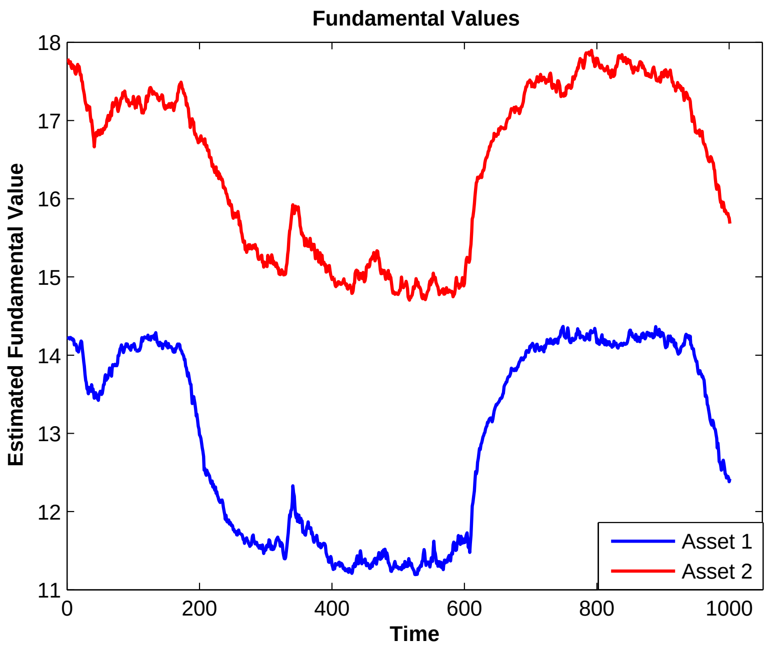


Figure 4.21: Estimation of Fundamental Values; Recessionary Scenario 2

that there is usually a prolonged recession which lasts long enough to activate the minimum consumption constraint. However, unlike the previous pattern, the economy manages to improve just before the fundamentalist is forced into bankruptcy. This makes a profound difference in the outcome when compared to the previous scenario. Because the trend follower does not gradually amass market wealth in the periods when his strategy is the most appropriate, the improvement in the economy leaves him very badly positioned for the forthcoming upswing in asset values.

Conversely, the fundamentalist, who has been acquiring cheap assets during the recessionary period, is in an excellent position to take advantage of improved market conditions. The common theme between the last two scenarios is that during recessionary periods it is never optimal to invest according to fundamental data (e.g. dividend yields). Whenever the trend follower is dominant, large inefficiencies and asset price bubbles can occur. On the other hand, the prevalence of the fundamentalist ensures prices are pushed back towards fundamental values and market efficiency is restored. This is illustrated in figures (4.21) through (4.26).

4.2 Challenges and Model Weaknesses

In the preceding sections gave graphical illustrations of the three most frequently-occurring scenarios within our model. We analyzed those qualitatively and were

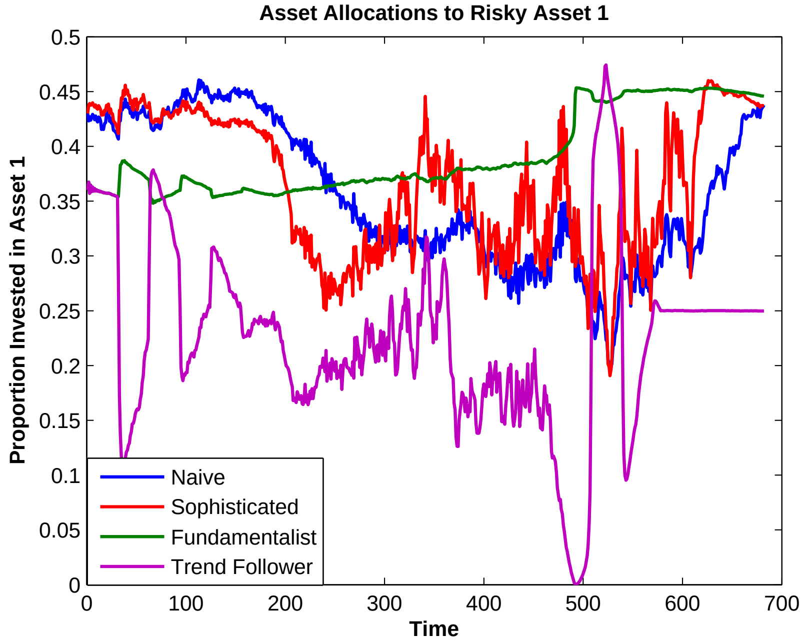


Figure 4.22: Asset Allocation to Risky Asset 1; Recessionary Scenario 2

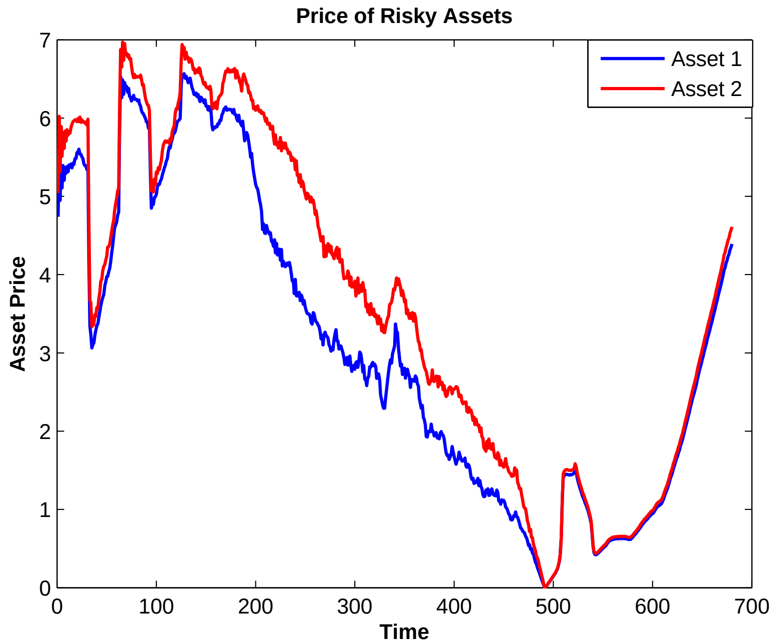


Figure 4.23: Asset Price Dynamics; Recessionary Scenario 2

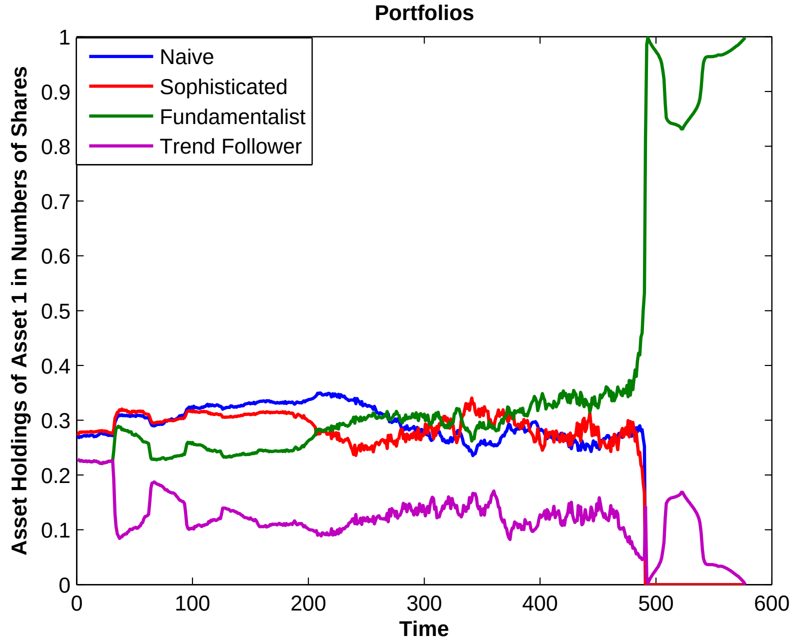


Figure 4.24: Asset Holdings of Risky Asset 1; Recessionary Scenario 2

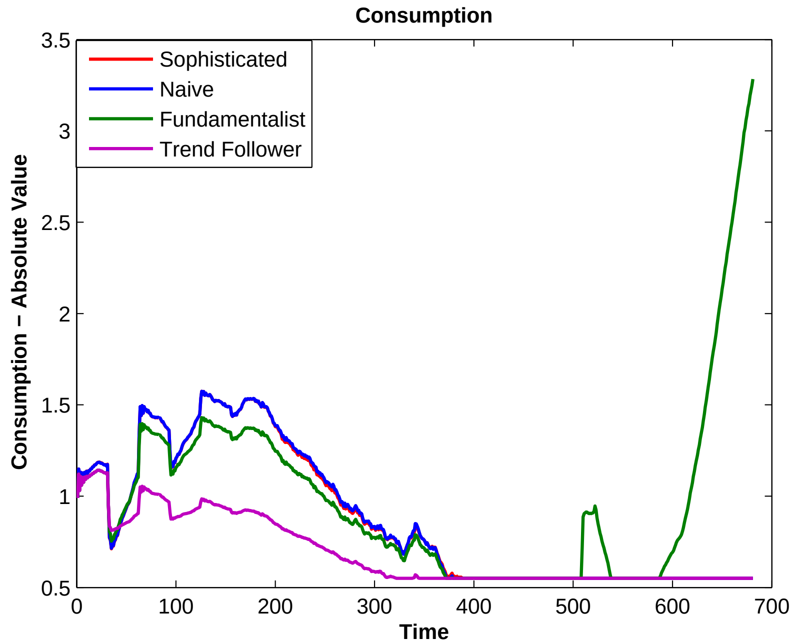


Figure 4.25: Consumption in Absolute Terms; Recessionary Scenario 2

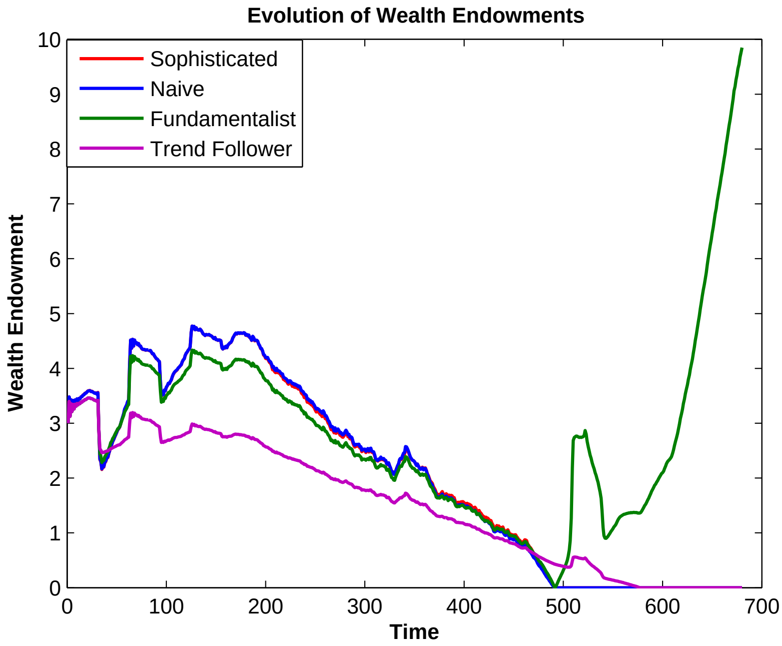


Figure 4.26: Evolution of Wealth Endowments; Recessionary Scenario 2

able to draw some conclusions about the desirability of each investment style under different market conditions. Unfortunately, however, the conclusions we obtained turned out to be fairly commonsense and easy to discern, even without the need for an agent-based or evolutionary finance model. This leaves a lot to be desired by the model.

Unfortunately, to address the ambitious list of interesting questions for potential investigation that we listed above has proven very difficult within the context of our model on account of a number of reasons. Some of these can be listed as follows:

- Dependence on initial conditions.
- Very strong dependence on parameter values.
- Difficulties in picking parameter values.
- Large number of parameters.
- Difficulties in model calibration to real-world data.
- The presence of unexpected feedback loops, cause by the model's endogenous price setting mechanism.

To provide a graphical illustration of the challenges we have faced, consider the following specific dividend scenario in figure (4.27) that we have picked on purpose.

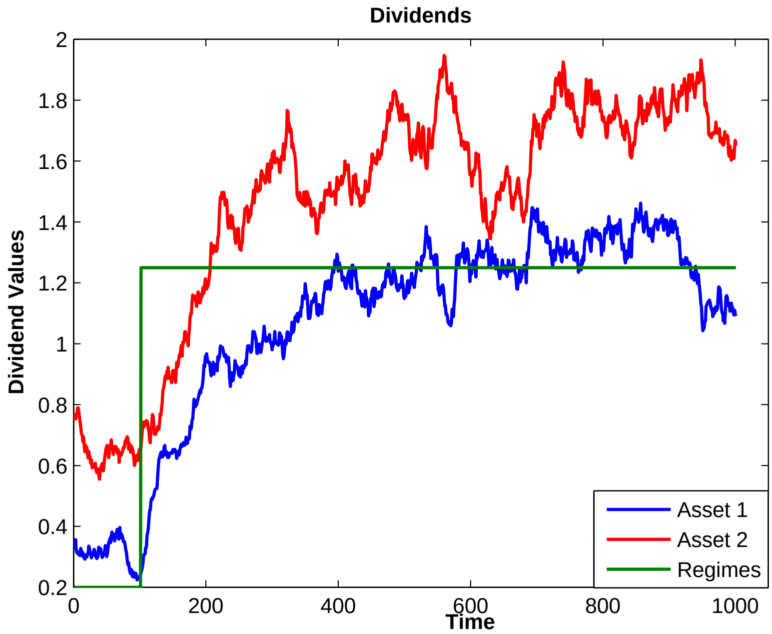


Figure 4.27: A Problematic Dividend Realization

At first sight there appears to be nothing spectacular about this scenario - an initial shock to the economy followed by a long period of prosperity. If, however, the parameters to the model are not picked exactly right, such long periods of prosperity, for example, can cause the system to become numerically unstable. In this example, we have picked the consumption parameter to be slightly less than what it needs to be in order to produce stable results. Because of the long, uninterrupted period of prosperity, agents are accumulating more and more wealth and through the endogenous price-setting mechanism, this gets passed on to asset prices and the wealth dynamic itself. Because investors take their investment decisions on the bases of asset prices and the size of their wealth endowments, such a case leads to progressively larger fluctuations in the agents' investment strategies, which in turn cause prices to fluctuate even more and so on. The dynamics gets locked in this cycle and gets progressively more destabilised. This is illustrated in figures (4.28) and (4.29).

Although we were able to explain this particular feedback loop, there is usually no way of controlling them by adjusting for them in advance, since during multiple simulations with random sample paths any such destabilising scenario may occur at any time. Additionally, note that the destabilisation of the system does not occur smoothly overtime. Rather, a sharp phase transition occurs, which makes this weakness even more difficult to control.

Apart from these unexpected feedback loops, the choice of parameters itself is often a challenge. In the sample runs above, we have specifically picked

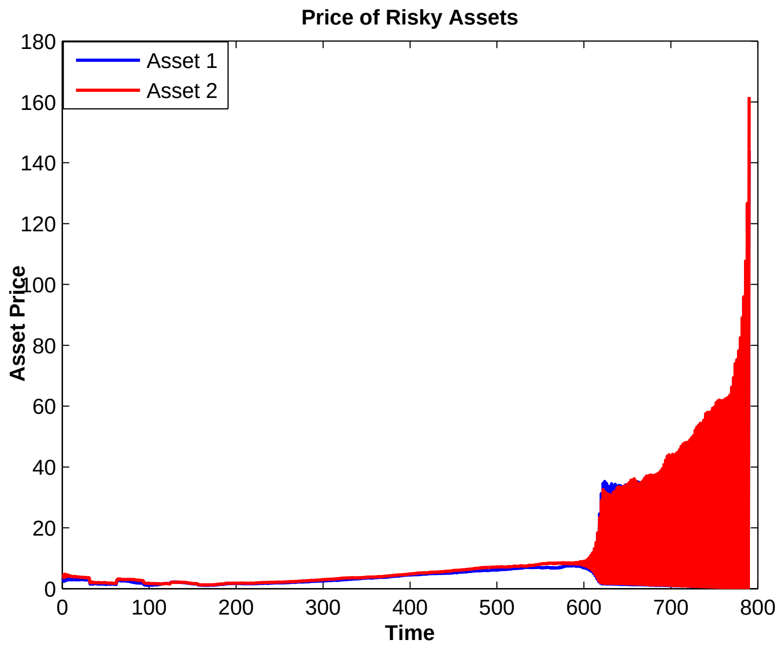


Figure 4.28: Destabilised Asset Price

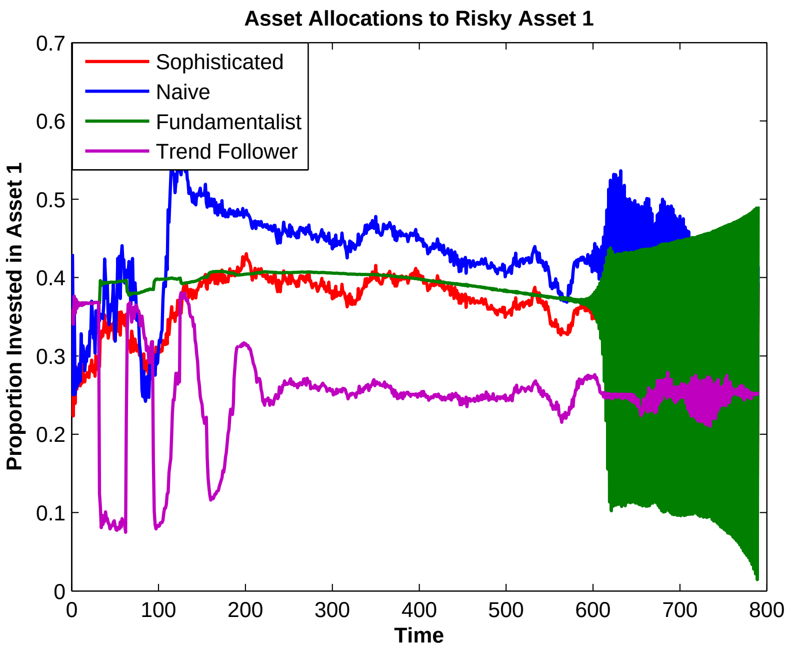


Figure 4.29: Destabilised Investment Strategies

parameters that keep the system stable for the required scenarios, however this was accomplished after a long process of trial and error. Bearing in mind the difficulties in picking parameters that are "just right", it becomes evident why it is very difficult to examine the sensitivity of the simulation results to changes in some of the parameters, like the consumption proportion or the minimum consumption constraint. Each such small perturbation destabilises the dynamics to an extent that comparison of results becomes a challenge.

Despite these difficulties, in the next section we propose an extension to the model by including two more investment strategies. This is done with the purpose of observing how popular investment approaches in finance, such as mean-variance optimisation, would behave within the context of the model. Nevertheless, one needs to bear in mind that all the above-mentioned difficulties with the model are also inherited by this model extension and even compounded by it, due to the larger number of parameters that need to be picked.

4.3 Benchmark Strategies

The results presented in the previous chapter were derived from an evolutionary finance model populated by four arbitrary types of investors. Even though we did our best in selecting what we believe are the most representative trading strategies in terms of the specific asset allocation problem we are facing, we admit that there are many more potential investment strategies that could also be included in the model. Therefore, it is not our objective to prescribe a single, ubiquitous investment approach to institutional investing under minimum consumption constraints, but rather to draw general conclusions as to what features of the different types of investment behaviour are selected by the market dynamics under different economic scenarios. More specifically, we were most concerned with the question "what type of investment behaviour minimizes the risk of bankruptcy during recessionary periods, while still retaining good investment performance under normal economic conditions"?

In order to examine the robustness of our general conclusions, in this section we add two more investment strategies to the model. The two new types of investment behaviour that we introduce are naïve diversification and mean-variance optimisation. These have been largely used in the finance literature as a benchmark for the comparison of results.

The addition of two new investor types requires a new adjustment of the model parameters, particularly the consumption proportion and the minimum guaranteed liability. This undoubtedly changes the outcome of the simulation experiments. However, the primary reason for the inclusion of the two benchmark strategies is to examine whether our general conclusions about the weak and strong points of different investment types still hold or the presence of these new investor types brings a qualitative change in the simulation results.

In the remainder of this chapter we first define the two new trading strategies and discuss some of the difficulties that may be encountered during their implementation within the context of our agent-based model.

4.3.1 Definition of Benchmark Strategies

4.3.1.1 Naïve Diversification

The first of the newly introduced investment types is a simple naïve diversification strategy, which requires that an investor split their wealth equally among all available assets, the risk-free asset included:

$$\lambda_k^N = \frac{1}{N_a}, \tag{4.1}$$

where N_a denotes the total number of available investment opportunities, including the risk-free asset. The naïve diversification strategy remains constant over time, and investors adopting such an approach would only rebalance their portfolios because of changes in asset prices and the size of their wealth endowments.

4.3.1.2 Mean-Variance Optimisation

The second benchmark strategy we will be using is a myopic mean-variance optimisation strategy. There are a number of ways to implement such an investment approach, however, unlike in classical finance, where quadratic utility is usually used, we will follow the agent-based and heterogeneous beliefs literature in specifying the investment strategy of a myopic mean-variance optimizer.

Unfortunately, the implementation of a mean-variance optimisation strategy in the agent-based literature is usually very restrictive. The norm is to provide demand functions for just a single risky asset. Since we would like to have a more general framework, capable of handling multiple risky assets, we propose the following approach to the implementation of mean-variance optimisation:

- Step 1: Treat the entire portfolio of all risky assets as a single risky asset and decide on the allocation between the risky portfolio and the risk-free asset. The criterion that will guide this choice is the maximization of the expected utility of consumption.

- Step 2: Once the proportion investable in risky assets is obtained, we solve for the proportions to be allocated to each risky asset based on the optimisation of the mean-variance criterion. As discussed previously, the possibility for the occurrence of insolvency and the structure of our proposed method for handling bankruptcy situations rule out short positions. Therefore, we proceed to solve the mean-variance optimisation problem by means of a quadratic programming algorithm.

4.3.1.3 Utility Maximization

Focusing our attention on the first step, in existing research on heterogeneous trading strategies, the starting point for specifying the behaviour of a mean-variance optimizer is usually the assumption of the constant absolute risk aversion (CARA) class of utility functions. A popular representative of the former is the negative exponential utility function of the following form:

$$U\left(C\right) = 1 - exp\left(-\gamma C\right), \quad \gamma > 0,$$

where $U\left(\cdot\right)$ denotes the utility function and C stands for consumption (see e.g. Ehrentreich (2008), Hommes & Wagener (2009), Anufriev & Dindo (2010)). Such a formulation for $U\left(C\right)$ satisfies the usual requirements for a utility function. Namely, it is increasing and concave, since:

$$U'\left(C\right) = \gamma exp\left(-\gamma C\right) > 0, \quad U''\left(C\right) = -\gamma^2 exp\left(-\gamma C\right) < 0.$$

In the expressions above γ denotes the Arrow-Pratt index of absolute risk aversion:

$$\gamma = -\frac{U''\left(C\right)}{U'\left(C\right)}.$$

Since consumption amounts are closely related to the uncertain returns an investor obtains from their portfolio in each time period, a frequently encountered assumption is that consumption follows a normal distribution; i.e. $C \sim N\left(\mu, \sigma^2\right)$. Under this assumption, the expected utility of consumption is given by:

$$E\left[U\left(C\right)\right] = \frac{1}{\sigma\sqrt{2\pi}} \int_{\infty}^{\infty} -exp - \left(\gamma C + \frac{\left(C-\mu\right)^2}{2\sigma^2}\right) dC.$$

Noting that

$$\gamma C + \frac{\left(C-\mu\right)^2}{2\sigma^2} = \frac{\left(C-\mu+\gamma\sigma^2\right)^2}{2\sigma^2} + \gamma\left(\mu - \frac{\gamma\sigma^2}{2}\right),$$

the above expectation reduces to

$$E\left[U\left(C\right)\right] = -exp - \left(\gamma\left(\mu - \frac{\gamma\sigma^2}{2}\right)\right). \tag{4.2}$$

In order to derive the desired asset allocations, the mean-variance optimizer maximizes expected utility. Since the exponential function is monotonic and strictly increasing, it is evident from the above equation that maximizing $\mu - \frac{\gamma\sigma^2}{2}$ will suffice. In other words, the mean-variance optimizer makes investment decisions by maximizing the mean-variance utility of next period's consumption, which in turn depends on the expected return of the portfolio as well as its dispersion. These two inputs characterize any efficient portfolio, however, and therefore the optimal portfolio choice for each investor is uniquely determined by their risk preferences, quantified by the risk-aversion coefficient γ.

Next period's consumption comes from the total return for that period from risky and risk-free assets. Therefore, maximizing (4.2) is equivalent to maximizing

$$\begin{aligned} E_{t_n} & \left[\lambda^{MV}_{p,t_n}\left(\left(\frac{D_{p,t_{n+1}}}{S_{p,t_n}}\right) + \left(\frac{S_{p,t_{n+1}}}{S_{p,t_n}} - 1\right)\right) + \left(1 - \lambda^{MV}_{p,t_n}\right) r\right] \\ - & \frac{\gamma}{2} Var_{t_n}\left[\lambda^{MV}_{p,t_n}\left(\left(\frac{D_{p,t_{n+1}}}{S_{p,t_n}}\right) + \left(\frac{S_{p,t_{n+1}}}{S_{p,t_n}} - 1\right)\right)\right], \end{aligned} \tag{4.3}$$

where E_{t_n} denotes the expectation at time t_n, λ^{MV}_{p,t_n} denotes the investment proportion of the mean-variance optimizer allocated to the risky portfolio at time t_n, and $D_{p,t_{n+1}}$ and S_{p,t_n} denote the combined dividend and asset price of the entire risky portfolio at times t_{n+1} and t_n respectively. As in the discussion above, γ denotes the risk aversion coefficient and Var_{t_n} refers to the variance of the risky portfolio calculated at time t_n.

After using standard results about the properties of the lognormal distribution and substituting for the components of portfolio return in equation (4.2) from (4.3), an application of first order conditions yields the mean-variance optimal investment proportion, which maximizes expected utility:

$$\lambda^{MV*}_{p,t_n} = \frac{E_{t_n}\left[\frac{D_{p,t_{n+1}}}{S_{p,t_n}} + \left(\frac{S_{p,t_{n+1}}}{S_{p,t_n}} - 1\right) - r\right]}{\gamma\sigma_p^2}, \tag{4.4}$$

where σ_p^2 denotes the variance of the risky portfolio at time t_n.

4.3.2 Quadratic Programming

With this allocation between the risky portfolio and the risk-free asset in mind, the next step is to obtain the investment proportions to each of the risky assets. Had there been no restrictions on investment behaviour, i.e. short sales allowed, this would have been a simple task. If short sales were allowed, then the results obtained in Merton (1972) would have been directly applicable to the problem at hand. However, because of the additional constraint of no short sales, we need to impose the requirement that investment proportions are nonnegative and solve the problem using a constrained optimisation formulation. Since the objective function to be optimised includes quadratic terms in the form of the risky assets' variances, we can use a wide variety of analytic and computational tools in order to solve a quadratic program of the form:

$$\min_{\lambda} \ \lambda_{t_n}^T \Sigma \lambda_{t_n}, \tag{4.5}$$

subject to

$$\begin{aligned}
\mu^T \lambda_{t_n} &= E_{t_n}\left[\frac{D_{p,t_{n+1}}}{S_{p,t_n}} + \left(\frac{S_{p,t_{n+1}}}{S_{p,t_n}} - 1\right)\right], && (4.6)\\
\underline{1}^T \lambda_{t_n} &= \lambda_{p,t_n}^{MV^*}, && (4.7)\\
\lambda_{t_n} &\geq 0. && (4.8)
\end{aligned}$$

In the above set of equations, λ_{t_n} is the vector of investment proportions to each risky asset at time t_n, Σ is the variance-covariance matrix of returns, μ is a vector of expected returns, and $\underline{1}$ denotes the vector, whose every element is 1. The quadratic program above requires the minimization of portfolio variance under the constraints that the risky portfolio's expected return should be equal to the expected return of the market portfolio, and that the investment proportions are nonnegative and sum up to the total amount available for risky investments. In keeping with the two fund separation theorem from classical finance, the quadratic program in equation (4.5) essentially attempts to find the point of tangency between the capital market line and the efficient portfolio frontier.

The quadratic program in equations (4.5) through (4.6) can be written more succinctly by dispensing with the two-step process outlined above and attempting to maximize the expected utility of consumption straight away, subject to the appropriate constraints. The quadratic program then becomes:

$$\max_{\lambda} \ \mu^T \lambda_{t_n} - \frac{\gamma}{2} \lambda_{t_n}^T \Sigma \lambda_{t_n}, \tag{4.9}$$

subject to

$$\begin{aligned}
1 - \underline{1}^T \lambda_{t_n} &\geq 0, && (4.10)\\
\lambda_{t_n} &\geq 0. && (4.11)
\end{aligned}$$

We will adopt this more direct approach in our implementation of the mean-variance optimisation strategy.

There exist a rich variety of approaches to solving the quadratic program (4.9). One of the earliest approaches, which forms a basis for more modern methods is based on the modification of the quadratic program to a linear program by using slack variables and applying the Karush-Kuhn-Tucker conditions (Kuhn & Tucker (1951)). The resulting linear program is then solved by means of the well-known simplex method (Wolfe (1959)). More modern and robust approaches also exist – Goldfarb and Idnani (1983), for instance, propose a dual method for solving strictly convex quadratic programs.

The main advantage of this approach is the fact that the minimum of the unconstrained objective function is used as a starting point. This is very convenient for problems where an initial primal feasible point is not easy to find. Such an approach helps to avoid having to find a solution for the phase I problem of the quadratic program. However, since for the mean-variance optimal portfolio problem, having one's entire wealth endowment invested in a single risky asset is always feasible, we could simply use this as an obvious starting point. Consequently, we choose to solve the quadratic program in equation (4.9) by means of the conceptually simpler primal active constraint set method (see e.g. Fletcher (2000), p. 240).

The basic premise of the active set method is to start with an initial feasible point and an active set of constraints. Progression to better feasible points is achieved by moving in the direction which optimizes the objective function and dropping constraints which are no longer active. If during the movement in a particular direction a constraint is encountered, then it enters the active set. Overall, the logic of the primal active set method resembles that of the simplex method, except that movement is not always between the vertices of the simplex.

The set of steps needed in order to implement the active set method can be summarised as follows:

1. Start at a feasible point λ_0 and an initial active set of constraints.

2. Calculate the optimum values for the variables λ^*_{EQP} and the Lagrangians ν^*_{EQP} of the corresponding equality constrained program defined by the current active set. Popular approaches for doing this are either the null space/QR method or KKT method (Karush-Kuhn-Tucker conditions). There are two possible outcomes of this step:

    - λ^*_{EQP} is feasible. Move to λ^*_{EQP} and check Lagrange multipliers. If all $\nu^*_{EQP} > 0$, then the solution is optimal, stop. Otherwise, remove a constraint with $\nu^*_{EQP} < 0$ from the active set of constraints.

    - λ^*_{EQP} is infeasible. Move as far as possible along the line segment between λ_0 and λ^*_{EQP} while staying feasible. Add to the active set the first encountered constraint that prevents further progress.

3. To find an initial feasible point and active set, solve the Phase I problem (not really necessary in the case of portfolio choice).

4. To avoid degeneracy, use Bland's anticycling rules.

The actual implementation of the above algorithm can be described by the following pseudocode:

1. Let $\lambda = \lambda_0$, $S = S_0$.

2. Let λ^*_{EQP} and ν^*_{EQP} solve the EQP in (3).

3. If $\lambda^*_{EQP} \in F$,
   If $\nu^*_{EQP} \geq 0$,
      stop: optimal.
   Else,
      remove from S a constraint with a negative ν^*_{EQP} entry.
      Set $\lambda = \lambda^*_{EQP}$.
      Go to (2).

   Else
   Let t_{max} solve

$$\max \ t$$

   subject to

$$\lambda + t\left(\lambda^*_{EQP} - \lambda\right).$$

   Set $\lambda = \lambda + t_{max}\left(\lambda^*_{EQP} - \lambda\right)$.
   Add to S one constraint that is binding at λ and that is violated by λ^*_{EQP}.
   Go to (2).

The full C++ implementation of the quadratic program in equation (4.9) is available in the accompanying CD.

4.3.3 Additional Challenges

It is evident both from (4.4) and (4.9) that the selection of a mean-variance optimal strategy depends on two market parameters – expected returns and variances. While the estimation of variances from historical data is a widely-accepted methodology in finance, the estimation of expected returns on the basis of past price history is more problematic due to the non-stationarity of financial time series. As an example, using the arithmetic mean of historical returns does not provide a sensible indication of the true expected return for a risky asset in cases where this mean is negative, since this clearly contradicts the level of expected return one might expect based on the asset's riskiness.

While in practice one solution to this problem is to estimate the expected return on a risky asset based on theoretical model, such as for example the capital asset pricing model, this is somewhat difficult to implement within the framework of our model since there is no proxy for the market portfolio. Other more subjective approaches, such as for instance the Black-Litterman model are also impractical since agents have no way of forming their own subjective expectations.

Since a more in-depth discussion about the estimation of expected returns is beyond the scope of this piece of research, we provide to ideas about possible solutions to this issue. The most obvious one is to follow the standard procedure in other agent-based papers, such as for instance Anufriev & Dindo (2010), and estimate next period's expected returns as the arithmetic mean of past returns over the course of some predetermined lookback period of length L:

$$\mu_{t_n}^k = \frac{1}{L}\sum_{\tau=1}^{L}\left[\frac{D_{t_{n+1}-\tau}^k}{S_{t_n-\tau}^k} + \left(\frac{S_{t_{n+1}-\tau}^k}{S_{t_n-\tau}^k} - 1\right)\right]. \tag{4.12}$$

In the absence of numerous risky assets and a proxy for the market portfolio, this choice is not as unreasonable as it might appear initially. When there are only two or three risky assets in the economy, whose dividend processes are determined by equations (1.14) or (1.16), they will be impacted by the same regime switches and will therefore be correlated. An application of the CAPM would then yield similar results to these in equation (4.12) as both the dividend processes and asset prices will be relatively strongly correlated.

The other choice is inspired by a different class of models that attempt to formulate an optimal investment policy by using a regime-switching mean-variance optimisation strategy. Such models, as presented in e.g. Ang & Bekaert (2002) use a dynamic optimisation procedure and capture the different regimes of the economy by observing correlation breakdowns in returns to suggest recessionary regimes.

While the model presented in Ang & Bekaert (2002) has other objectives in mind, we draw inspiration from it and try to incorporate learning about the regimes of the economy within the mean-variance optimisation strategy. To accomplish this we use the Bayesian updating conditional probabilities results. The expected return can be decomposed into an expected dividend yield component and an expected capital gains part. For the expected capital gains we continue to use historical data as shown in equation (4.12). For the expected dividend yield component, though, we can use the conditional probabilities of being in each regime to weight the mean reverting dividend levels in each regime in order to form an expectation of the future dividend yield. In other words, we can use:

$$E_{t_n}\left(\frac{D_{p,t_{n+1}}}{S_{p,t_n}}\right) = \frac{P_{up}\overline{\mu} + (1 - P_{up})\,\underline{\mu}}{S_{p,t_n}}. \tag{4.13}$$

This formulation, however, doesn't produce a significant difference in the dynamics since one of the strategies presented above - the sophisticated dividend yield investor - already uses a similar characterisation.

With regard to the variances of returns, the most sensible and straightforward choice for the context of our model is to use historical volatilities:

$$\sigma_{t_n}^{k^2} = \frac{1}{L-1}\sum_{\tau=1}^{L}\left(\left[\frac{D_{t_{n+1-\tau}}^{k}}{S_{t_{n-\tau}}^{k}} + \left(\frac{S_{t_{n+1-\tau}}^{k}}{S_{t_{n-\tau}}^{k}} - 1\right)\right] - \mu_{t_n}^{k}\right)^2. \tag{4.14}$$

The last interesting point worth mentioning about the implementation of a mean-variance optimisation strategy within the context of our model is that while both quadratic programs (4.5) and (4.9) consider the no short sales constraint, this requirement is actually too weak. Recall that theorem (1.1.1) demands investment proportions to be not simply nonnegative but strictly positive. If this condition is violated, then invertibility cannot be guaranteed and therefore we cannot write an explicit solution to the wealth dynamics. Hence, the requirement of full diversification, i.e. $\lambda_{t_n}^k > 0$ for all k and n, is of vital importance. Since the set of constraints related to the quadratic program (4.9) are too weak to guarantee this, we take further action to ensure that investment proportions are strictly positive.

In the rare cases when the allocation to some of the risky assets turns out to be exactly zero, we artificially increase it by making it arbitrarily small (e.g. 1% of the investor's total wealth endowment) and simultaneously decrease the proportions allocated to the other risky assets by equal amounts, so that the budget constraint $\sum_{k=0}^{K}\lambda_{t_n}^k = 1$ is not violated. While the resulting portfolio will not be, strictly speaking, mean-variance optimal, it will not deviate by much. However, at the same time it will ensure that the full diversification assumption is not violated and the wealth dynamics can be solved.

In this section we described some extensions to the basic model discussed in the previous two chapters. We introduced two new popular investment strategies to act as a benchmark for comparison purposes – naïve diversification and mean-variance optimisation. While the former presents no additional challenges, the implementation of the latter gives rise to some technical difficulties caused by the structure of our basic model.

In the end, neither of these benchmark strategies changes materially the qualitative conclusions about the surviving strategies in the three most frequently encountered scenarios discussed above. The inclusion of these strategies, however, introduces additional parameters that need to be set with some degree of precision and thus increases the potential for destabilisation of the system. With this in mind, we conclude that any marginal benefits of including these two benchmanr strategies is not significant when juxtaposed against the potential for additional destabilisation of the system.

Chapter 5 Conclusion

In this thesis we presented an evolutionary finance model of institutional investment policies with minimum consumption constraints. Our methodology was heavily influenced by recent advances in the fields of evolutionary finance and agent-based modelling. The basis of our approach was to address the problem of optimal asset allocation with minimum consumption constraints from the perspective of market selection of investment strategies.

To this end we specified an artificial financial market populated by a number of agents following predetermined strategies. These strategies were specified as model primitives and were not obtained by an analytical procedure, such as utility maximisation. We let the market evolve and select the most successful strategy. Success within the context of our model was measured by the size of an agent's wealth endowment.

The model extended previous work in evolutionary finance by introducing an explicit minimum consumption threshold, which opened the possibility for bankruptcy. We investigated a number of ways to handle this occurrence, as well as its potential consequences for the market dynamics.

We solved the proposed market dynamics by means of numerical simulations. Qualitative analysis of the results illustrated three main scenarios that typically occur within our specification of the model. The most typical pattern of scenarios were those characterised by favourable economic conditions. Under these scenarios, few bankruptcies were observed. Investment styles that remain invested in risky assets for longer periods of time tend to outperform more erratic strategies, such as trend-following. Cases of market instability, excess volatility and deflationary spirals were rare in those scenarios.

The remaining two scenarios both featured pronounced and long-lasting recessionary periods. In the scenarios that featured severe and prolonged recessions with little recovery, the trend-following strategy emerged preferable. Fire sale conditions were particularly observable when all investors started consuming at the minimum threshold level. An interesting feature was that the trend-following strategy did not outperform the others gradually, but by means of a sharp phase transition. A large amount of instability, uncertainty and severe deflationary spirals were observed during these scenarios.

The third group of scenarios that emerged also features recessionary periods, which however, although severe, managed to recover just before the fundamentalist investor type was forced into bankruptcy. Because of the stabilising effect on the market that fundamentalist investment strategies have, this group of scenarios was characterised by healthy and efficient markets, where deflationary spirals are unlikely and where an asset's price eventually reverts back to its fundamental value. An interesting observation was that under no scenario were dividend yield or mean-variance optimising strategies selected by the market.

We proposed the model specification in this thesis with the ultimate objective of gaining deeper insight into some of the most widely-spread asset-liability problems that pension funds and insurance companies face. However, we discovered that our model formulation suffers from a number of deficiencies, which can hinder the accomplishment of this goal. Ultimately, a new and much leaner model specification will be necessary in order to be able to calibrate it to real-world data and draw useful investment policy implications for large institutional investors.

Chapter 6 Appendix

6.1 Proof of Theorem 3.1.1

In this section we give a proof to Theorem 3.1.1. The outline of the proof follows closely the analysis in Palczewski & Schenk-Hoppé (2010a), Appendix E, with a few minor modifications. Since we are interested in proving the existence and uniqueness of the solution of a discrete-time dynamics, we will not discuss the parts of the proof relevant for the reduced-dimension dynamics as well as its continuous counterpart.

Proof Recall the semi-explicit form of the dynamics:

$$\left[\mathrm{Id} - \Theta\left(\Lambda_{t_n}, V_{t_n}\right)\Lambda_{t_{n+1}}\right] V_{t_{n+1}} = M_{t_{n+1}} + \Theta\left(\Lambda_{t_n}, V_{t_n}\right) D_{t_{n+1}} - C\left(V_{t_n}\right), \quad (6.1)$$

where Id denotes an $I \times I$ identity matrix. Let $A = \left[Id - \Theta\left(\Lambda_{t_n}, V_{t_n}\right)\Lambda_{t_{n+1}}\right]$. A necessary and sufficient condition for the dynamics (6.1) to be well-defined is if matrix $A \in \mathbb{R}^{I\times I}$ is invertible. The proof of A's invertibility proceeds as follows.

Firstly, we note that all column sums of $\Lambda_{t_{n+1}} \in \mathbb{R}^{K\times I}$ are strictly less than one due to the assumption of fully diversified strategies: $\sum_{k=1}^{K} \Lambda_{t_{n+1},ki} < 1$. Secondly, since the risky assets have a net positive supply of one and the market for the risky assets clears, all column sums of $\Theta\left(\Lambda_{t_n}, V_{t_n}\right)$ are equal to one: $\sum_{i=1}^{I} \Theta\left(\Lambda_{t_n}, V_{t_n}\right)_{ik} = 1$.

Bearing in mind the two facts mentioned above, it turns out that matrix A has a column-dominant diagonal. Each diagonal entry strictly dominates the sum of absolute values of the remaining entries in the corresponding column:

$$A_{ii} > \sum_{j=1, j\neq i}^{I} |A_{ji}|, \quad i = 1, \ldots, I. \quad (6.2)$$

This claim is easy to verify. The elements of matrix A are given by:

$$\mathbf{1}_{i=j} - \sum_{k=1}^{K} \Theta\left(\Lambda_{t_n}, V_{t_n}\right)_{ik} \Lambda_{t_{n+1},ki}.$$

All off-diagonal entries are non-positive and the diagonal entries are non-negative. The condition in equation (6.2) can be restated as:

$$1 - \sum_{k=1}^{K} \Theta\left(\Lambda_{t_n}, V_{t_n}\right)_{ik} \Lambda_{t_{n+1},ki} > \sum_{j=1,j\neq i}^{I} \sum_{k=1}^{K} \Theta\left(\Lambda_{t_n}, V_{t_n}\right)_{ik} \Lambda_{t_{n+1},ki}$$

$$1 > \sum_{j=1,j\neq i}^{I} \sum_{k=1}^{K} \Theta\left(\Lambda_{t_n}, V_{t_n}\right)_{ik} \Lambda_{t_{n+1},ki} + \sum_{k=1}^{K} \Theta\left(\Lambda_{t_n}, V_{t_n}\right)_{ik} \Lambda_{t_{n+1},ki}$$

$$\sum_{i=1}^{I} \sum_{k=1}^{K} \Lambda_{t_{n+1},ki} \Theta\left(\Lambda_{t_n}, V_{t_n}\right)_{ik} < 1.$$

The inequality is proved by taking advantage of the properties of the sums of portfolios and investment strategies shown above, namely:

$$\begin{aligned} \sum_{k=1}^{K} \sum_{i=1}^{I} \Lambda_{t_{n+1},ki} \Theta\left(\Lambda_{t_n}, V_{t_n}\right)_{ik} &= \sum_{i=1}^{I} \left(\sum_{k=1}^{K} \Lambda_{t_{n+1},ki} \right) \Theta\left(\Lambda_{t_n}, V_{t_n}\right)_{ik} \\ &< \sum_{i=1}^{I} \Theta\left(\Lambda_{t_n}, V_{t_n}\right)_{ik} = 1. \end{aligned} \quad (6.3)$$

The result in equation (6.3) proves the result in equation (6.2). The latter proves that matrix A is invertible and A^{-1} maps the non-negative orthant to itself (see Murata (1977), p. 24, Theorem 23). ∎

6.2 Proof of Lemma 3.4.1

Proof From Equation (3.25) we have:

$$X_{k+1} X'_{k+1} = A X_k \left(A X_k\right)' + A X_k V'_{k+1} + V_{k+1} \left(A X_k\right)' + V_{k+1} V'_{k+1}.$$

However, X_{k+1} is a vector with only one element equal to 1. Multiplying this vector by its transpose gives a $N \times N$ matrix of zeros with only one entry equal to one. Or in other words, a diagonal matrix with X_{k+1} on its main diagonal:

$$X_{k+1} X'_{k+1} = diag\left(X_{k+1}\right) = diag\left(A X_k\right) + diag\left(V_{k+1}\right).$$

The result follows.

The proof follows from the fact that all terms on the RHS involving V_{k+1} are zero-mean martingale increments, as was already proven. Therefore, taking expectations and conditioning on X_k, only the first and the third term remain non-zero:

$$\langle V_{k+1} \rangle = E\left[V_{k+1} V'_{k+1} | X_k\right] = diag\left(A X_k\right) - A \; diag\left(X_k\right) A'. \quad ∎$$

6.3 Proof of Lemma 3.4.2

Proof Representing λ_k in terms of a sum rather than a product and using the definition of c^i_{k+1} yields:

$$\begin{aligned}
E\left[\lambda_{k+1}|\Xi_k\right] &= E\left[\prod_{i=1}^{M}\left(\frac{1}{Mc^i_{k+1}}\right)^{Y^i_{k+1}}|\middle| Xi_k\right] \\
&= E\left[\sum_{i=1}^{M}\frac{1}{Mc^i_{k+1}}Y^i_{k+1}|\middle| Xi_k\right] \\
&= \frac{1}{M}\sum_{i=1}^{M}\frac{1}{c^i_{k+1}}P\left(Y^i_{k+1}=1|\Xi_k\right) \\
&= \frac{1}{M}\sum_{i=1}^{M}\frac{1}{c^i_{k+1}}c^i_{k+1} = 1.
\end{aligned}$$

In getting from row two to row three we used a special case of Fubini's theorem (see e.g. Ash & Doléans-Dade, 2000, p.108) we can interchange expectations and summations. The summation corresponds to the counting measure, whereas the expectation $E\left[\cdot\right]$ is taken under the original probability measure P. ∎

6.4 Proof of Theorem 3.4.3

Proof Let B be any set in Ξ. It must be shown that:

$$\int_B \overline{E}\left[\phi|\Xi\right]d\overline{P} = \int_B \frac{E\left[\Lambda\phi|\Xi\right]}{E\left[\Lambda|\Xi\right]}d\overline{P}.$$

Define:

$$\psi\begin{cases}\frac{E\left[\Lambda\phi|\Xi\right]}{E\left[\Lambda|\Xi\right]} & \text{if } E\left[\Lambda|\Xi\right]>0,\\ 0 & \text{otherwise.}\end{cases}$$

In other words, we have to show that for any set A in Ξ:

$$\int_A \overline{E}\left[\phi|\Xi\right]d\overline{P} = \int_A \psi d\overline{P}.$$

Write:

$$G = \left\{w : E\left[\Lambda|\Xi\right]=0\right\},$$

so that $G \subset \Xi$. Then:

$$\int_G E\left[\Lambda|\Xi\right]dP = 0 = \int_G \Lambda dP,$$

since by the law of iterated expectations, conditional expectations with respect to a sub-σ-algebra are random variables on the whole probability space. Also, by the definition of Λ, $\Lambda \geq 0$ almost surely. Thus, we have two possibilities: either $P(G) = 0$, or the restriction of Λ to G is 0 almost surely. But by hypothesis, $\overline{P}$ is absolutely continuous with respect to P, which means that $\overline{P}(G) = 0$ whenever $P(G) = 0$. So, in either case, $\Lambda = 0$ almost surely on G.

Since $\Lambda \geq 0$ by definition, denote the complement of set G as:

$$G^c = \{w : E[\Lambda|\Xi] > 0\}.$$

For an arbitrary set A, suppose $A \in \Xi$. Then A can be represented as $A = B \cup C$, where $B = A \cap G^c$ and $C = A \cap G$. Then:

$$\begin{aligned}\int_A \overline{E}[\phi|\Xi]\, d\overline{P} &= \int_A \phi d\overline{P} = \int_A \phi\Lambda dP \\ &= \int_B \phi\Lambda dP + \int_C \phi\Lambda dP, \end{aligned} \tag{6.4}$$

where the first equality comes from the fact that conditional expectations with respect to a sub-σ-algebra are random variables with respect to the entire σ-algebra. Since $\Lambda = 0$ almost surely on $C \subset G$:

$$\int_C \phi\Lambda dP = 0 = \int_C \psi d\overline{P}, \tag{6.5}$$

by the definition of ψ.

Concentrating on the set B, we have:

$$\begin{aligned}\int_B \psi d\overline{P} &= \int_B \frac{E[\Lambda\phi|\Xi]}{E[\Lambda|\Xi]} d\overline{P} \\ &= \overline{E}\left[I_B \frac{E[\Lambda\phi|\Xi]}{E[\Lambda|\Xi]}\right] \\ &= E\left[I_B \Lambda \frac{E[\Lambda\phi|\Xi]}{E[\Lambda|\Xi]}\right] \\ &= E\left[E\left[I_B \Lambda \left. \frac{E[\Lambda\phi|\Xi]}{E[\Lambda|\Xi]} \right| \Xi\right]\right] \\ &= E\left[I_B E[\Lambda|\Xi] \frac{E[\Lambda\phi|\Xi]}{E[\Lambda|\Xi]}\right] \\ &= E[I_B E[\Lambda\phi|\Xi]] \\ &= E[I_B \Lambda\phi],\end{aligned}$$

where the fifth equality follows from the fact that the indicator function I_B and the ratio $\frac{E[\Lambda\phi|\Xi]}{E[\Lambda|\Xi]}$ are constants and can be taken out of the expectation operator. That is:

$$\int_B \Lambda\phi dP = \int_B \psi d\overline{P}. \tag{6.6}$$

Using the result in equation (6.4) and adding equations (6.5) and (6.6), we have:

$$\int_C \Lambda\phi dP + \int_B \Lambda\phi dP = \int_A \Lambda\phi dP$$

$$= \int_A \overline{E}\left[\phi|\Xi\right] d\overline{P} = \int_A \psi d\overline{P},$$

and the result follows. ∎

6.5 Proof of Lemma 3.4.5

Proof

$$\begin{aligned}
\overline{P}\left(Y^j_{k+1} = 1|\Xi_k\right) &= \overline{E}\left[\langle Y_{k+1}, f_j\rangle|\Xi_k\right] \\
&= \frac{E\left[\Lambda_{k+1}\langle Y_{k+1}, f_j\rangle|\Xi_k\right]}{E\left[\Lambda_{k+1}|\Xi_k\right]} \\
&= \frac{\Lambda_k E\left[\lambda_{k+1}\langle Y_{k+1}, f_j\rangle|\Xi_k\right]}{\Lambda_k E\left[\lambda_{k+1}|\Xi_k\right]} \\
&= E\left[\lambda_{k+1}\langle Y_{k+1}, f_j\rangle|\Xi_k\right] \\
&= E\left[\left.\prod_{i=1}^{M}\left(\frac{1}{Mc^i_{k+1}}\right)^{Y^i_{k+1}} \langle Y_{k+1}, f_j\rangle\right| \Xi_k\right] \\
&= E\left[\left.\sum_{i=1}^{M}\left(\frac{1}{Mc^i_{k+1}}\right) Y^i_{k+1} Y^j_{k+1}\right| \Xi_k\right] \\
&= \frac{1}{Mc^j_{k+1}} E\left[Y^j_{k+1}|\Xi_k\right] \\
&= \frac{1}{Mc^j_{k+1}} c^j_{k+1} = \frac{1}{M} = \overline{P}\left(Y^j_{k+1} = 1\right).
\end{aligned}$$

This is a quantity independent of Ξ_k, which is the required result. The first equality follows from property (1.21), the second one follows from lemma (1.4.4). The third equality uses the fact that Λ_k is Ξ_k-measurable and is therefore a constant. The fourth equality follows from lemma (1.4.2). The next two equalities use property (1.38) and the two different representations of λ_{k+1}. The seventh equality uses the fact that since only one element of Y^i_{k+1} is one, while all the others are zero, the sum $\sum_{i=1}^{M} Y^i_{k+1}$ equals one and the term corresponding to the element which is one – $\frac{1}{Mc^j_{k+1}}$ – is a constant when conditioned on Ξ_k. The last equality follows from the definition of c^j_{k+1} in equation (1.30). ∎

6.6 Proof of Lemma 3.4.6

Proof Using properties (1.21) and (1.38), and following the proof of lemma (1.4.2), we have:

$$\begin{aligned}
\overline{E}\left[\overline{\lambda}_{k+1}|\Xi_k\right] &= \overline{E}\left[\prod_{i=1}^{M}\left(Mc_{k+1}^i\right)^{Y_{k+1}^i}\middle|\Xi_k\right] \\
&= \overline{E}\left[\sum_{i=1}^{M}\left(Mc_{k+1}^i\right)Y_{k+1}^i\middle|\Xi_k\right] \\
&= M\sum_{i=1}^{M}c_{k+1}^i\overline{P}\left(Y_{k+1}^i=1|\Xi_k\right) \\
&= M\sum_{i=1}^{M}\frac{c_{k+1}^i}{M}=\sum_{i=1}^{M}c_{k+1}^i=1.
\end{aligned}$$

The first two equalities show different representations of $\overline{\lambda}_{k+1}$ and follow from property (1.38). The third equality comes from the equivalence of expectations and probabilities for simple random variables demonstrated in property (1.21). The last equality follows from the definition of c_{k+1}^i and the fact that under $\overline{P}$, $\{Y_k\}$ are a sequence of i.i.d random variables, each with a uniform distribution over the M elements of its range space. ∎

6.7 Proof of Lemma 3.4.7

Proof The proof proceeds in a way similar to the proof of lemma (1.4.5). Using lemmas (1.4.4) and (1.4.6), as well as properties (1.21) and (1.38), we have:

$$\begin{aligned}
P\left(Y_{k+1}^j=1|\Xi_k\right) &= E\left[\langle Y_{k+1},f_j\rangle|\Xi_k\right] \\
&= \frac{\overline{E}\left[\overline{\Lambda}_{k+1}\langle Y_{k+1},f_j\rangle|\Xi_k\right]}{\overline{E}\left[\overline{\Lambda}_{k+1}|\Xi_k\right]} \\
&= \frac{\overline{\Lambda}_k\overline{E}\left[\overline{\lambda}_{k+1}\langle Y_{k+1},f_j\rangle|\Xi_k\right]}{\overline{\Lambda}_k\overline{E}\left[\overline{\lambda}_{k+1}|\Xi_k\right]} \\
&= \overline{E}\left[\overline{\lambda}_{k+1}\langle Y_{k+1},f_j\rangle|\Xi_k\right] \\
&= \overline{E}\left[\prod_{i=1}^{M}\left(Mc_{k+1}^i\right)^{Y_{k+1}^i}\langle Y_{k+1},f_j\rangle\middle|\Xi_k\right] \\
&= \overline{E}\left[\sum_{i=1}^{M}\left(Mc_{k+1}^i\right)Y_{k+1}^iY_{k+1}^j\middle|\Xi_k\right] \\
&= Mc_{k+1}^j\overline{E}\left[Y_{k+1}^j|\Xi_k\right] \\
&= Mc_{k+1}^j\overline{P}\left(Y_{k+1}^j=1|\Xi_k\right) \\
&= \frac{Mc_{k+1}^j}{M}=c_{k+1}^j.
\end{aligned}$$

The only two notable differences from the proof of lemma (1.4.5) are that the fourth equality now follows from lemma (1.4.6) instead of lemma (1.4.2) and that the last equality follows from the fact that under $\overline{P}$, $\{Y_k\}$ is a sequence of i.i.d. random variables, each distributed uniformly over the M elements of its range space. ∎

6.8 Proof of Theorem 3.4.8

Proof The proof of theorem (1.4.8), as most of the proofs above, uses properties (1.21) and (1.38). Additionally, it relies on the independence assumptions under $\overline{P}$, the mutual independence of V_{k+1} and W_{k+1} under P, as well as the fact that $\sum_{j=1}^{N}\langle X_k, e_j\rangle = 1$. Starting from the definition of $q_{k+1}(e_r)$, we have:

$$\begin{aligned}
q_{k+1}(e_r) &= \overline{E}\left[\overline{\Lambda}_{k+1}\langle X_{k+1}, e_r\rangle \middle| \Upsilon_{k+1}\right] \\
&= \overline{E}\left[\langle AX_k + V_{k+1}, e_r\rangle \overline{\Lambda}_k \prod_{i=1}^{M}\left(Mc_{k+1}^{i}\right)^{Y_{k+1}^{i}} \middle| \Upsilon_{k+1}\right] \\
&= M\overline{E}\left[\langle AX_k, e_r\rangle \overline{\Lambda}_k \prod_{i=1}^{M}\left(\langle CX_k, f_i\rangle\right)^{Y_{k+1}^{i}} \middle| \Upsilon_{k+1}\right] \\
&= M\sum_{j=1}^{N}\overline{E}\left[\langle X_k, e_j\rangle a_{rj}\overline{\Lambda}_k | \Upsilon_{k+1}\right]\prod_{i=1}^{M} c_{ij}^{Y_{k+1}^{i}} \\
&= M\sum_{j=1}^{N}\overline{E}\left[\langle X_k, e_j\rangle a_{rj}\overline{\Lambda}_k | \Upsilon_{k}\right]\prod_{i=1}^{M} c_{ij}^{Y_{k+1}^{i}} \\
&= M\sum_{j=1}^{N} q_k(e_j)\, a_{rj}\prod_{i=1}^{M} c_{ij}^{Y_{k+1}^{i}}.
\end{aligned}$$

The second equality above follows from the definitions of X_{k+1} and $\overline{\Lambda}_{k+1}$. The third equality uses result (1.46), which states that V_{k+1} is a zero-mean martingale increment when conditioned on Υ_{k+1}.

The fourth equality follows from an algebraic manipulation. Instead of multiplying the element of X_{k+1} that corresponds to state r by $\overline{\Lambda}_k$, we take each state of X_k, multiply it by the transition probability of moving to state r in the next period, and sum them over all possible states of X_k. The same representation is used for the conditional probability of the observation process $\prod_{i=1}^{M}\left(\langle CX_k, f_i\rangle\right)^{Y_{k+1}^{i}}$ – the conditional probability of the observation process is written as the sum of the probabilities of observing the realized value of Y_{k+1} on condition that X_k was in each of its possible N states. Note that in this step of the proof property (1.21) was used in order to exchange expectations and probabilities. Furthermore, changing the order of the expectation and the summation is allowed under Fubini's theorem.

The fifth equality is a simple application of the fact that under $\overline{P}$, $\{Y_k\}$ are i.i.d. random variables and therefore conditioning on Υ_k instead of Υ_{k+1} will not change the conditional distribution in the fourth equality.

The last equality is obtained using the definition of $q_k(e_j)$ and the result follows from using the notation introduced just prior to theorem (1.4.8). ∎

6.9 Proof of Theorem 3.4.9

Proof The proof of this theorem relies mostly on Lemma (1.4.1) and properties (1.21) and (1.38), as well as the assumption of independence under the probability measure $\overline{P}$.

$$\begin{aligned}
&\gamma_{k+1,k+1}(H_{k+1}) \\
=\;& \overline{E}\left[X_{k+1}H_{k+1}\overline{\Lambda}_{k+1}\,|\Upsilon_{k+1}\right] \\
=\;& \overline{E}\left[(AX_k+V_{k+1})\left(H_k+\alpha_{k+1}+\langle\beta_{k+1},V_{k+1}\rangle+\langle\delta_{k+1},Y_{k+1}\rangle\right)\overline{\Lambda}_k\overline{\lambda}_{k+1}\,|\Upsilon_{k+1}\right] \\
=\;& \overline{E}\left[\left(\left(H_k+\alpha_{k+1}+\langle\delta_{k+1},Y_{k+1}\rangle\right)AX_k+\langle V_{k+1}\rangle\beta_{k+1}\right)\overline{\Lambda}_k\overline{\lambda}_{k+1}\,|\Upsilon_{k+1}\right] \\
=\;& \sum_{j=1}^{N}c_j(Y_{k+1})\,\overline{E}\left[\left(\left(H_k+\alpha_{k+1}+\langle\delta_{k+1},Y_{k+1}\rangle\right)a_j+\langle V_{k+1}\rangle\beta_{k+1}\right)\overline{\Lambda}_k\langle X_k,e_j\rangle\,|\Upsilon_{k+1}\right].
\end{aligned}$$

Again, since Y are i.i.d. random variables under the probability measure $\overline{P}$, conditioning on Υ_{k+1} gives the same result as conditioning on Υ_k. Using Lemma (1.4.1) and substituting for the newly-introduced notation yields the required result.

In the above proof, the first equality simply uses the definition of $\gamma_{k+1,k+1}(H_{k+1})$. The second equality follows from substituting for the values of X_{k+1}, H_{k+1} and $\overline{\Lambda}_{k+1}$. The third equality follows from the fact that V_{k+1} is a zero-mean martingale increment when conditioned on Σ_k as well as from Lemma (1.4.1). As was the case in Lemma (1.4.1), we have:

$$\begin{aligned}
\overline{E}\left[V_kV_k'\,|\Upsilon_k\right] &= \overline{E}\left[\overline{E}\left[V_kV_k'\,|X_0,X_1,\ldots,X_k,\Upsilon_k\right]|\Upsilon_k\right] \\
&= \overline{E}\left[\langle V_k\rangle\,|\Upsilon_k\right].
\end{aligned}$$

The last equality follows from properties (1.21) and (1.38), as well as Fubini's theorem, which allows the exchange of the summation and expectation operators. ∎

6.10 Proof of Theorem 3.4.10

Proof Similarly to the model with discrete observations, V_{k+1} is a zero-mean Σ_k-martingale increment, i.e. $E\left[V_{k+1}\,|\Sigma_k\right]=0$. Also, note that since the w_k are

$N(0,1)$ i.i.d. random variables, they do not bring any new information when added to a conditioning set. Therefore:

$$E\left[V_{k+1}\,|\Xi_k, w_{k+1}\right] = E\left[V_{k+1}\,|\Sigma_k\right] = 0.$$

Consequently, we can write :

$$E\left[V_{k+1}\,|\Upsilon_{k+1}\right] = E\left[E\left[V_{k+1}\,|\Xi_k, w_{k+1}\right]|\Upsilon_{k+1}\right] = 0$$

and

$$\begin{aligned}
\hat{X}_{k+1} &= E\left[X_{k+1}\,|\Upsilon_{k+1}\right] = E\left[AX_k + V_{k+1}\,|\Upsilon_{k+1}\right] \\
&= AE\left[X_k\,|\Upsilon_{k+1}\right]. \qquad (6.7)
\end{aligned}$$

Substituting equation (1.60) for $E\left[X_k\,|\Upsilon_{k+1}\right]$ yields the final result. ∎

6.11 Proof of Lemma 3.4.11

Proof First note, that $\overline{P}\left(y_{k+1} \leq t\,|\Xi_k\right) = \overline{E}\left[I_{y_{k+1}\leq t}\,|\Xi_k\right]$. Using the conditional Bayes theorem (see Lemma (1.4.4)), this is:

$$\begin{aligned}
&= \frac{E\left[\Lambda_{k+1} I_{y_{k+1}\leq t}\,|\Xi_k\right]}{E\left[\Lambda_{k+1}\,|\Xi_k\right]} \\
&= \frac{\Lambda_k}{\Lambda_k}\frac{E\left[\lambda_{k+1} I_{y_{k+1}\leq t}\,|\Xi_k\right]}{E\left[\lambda_{k+1}\,|\Xi_k\right]}.
\end{aligned}$$

Recall that under P, $w_{k+1} = \frac{y_{k+1} - \langle c, X_k\rangle}{\langle \sigma, X_k\rangle}$. Now, consider the denominator:

$$\begin{aligned}
E\left[\lambda_{k+1}\,|\Xi_k\right] &= E\left[E\left[\lambda_{k+1}\,|\Xi_k, X_{k+1}\right]|\Xi_k\right] \\
&= \int_{-\infty}^{\infty} \frac{\langle \sigma, X_{k+1}\rangle \phi\left(y_{k+1}\right)}{\phi\left(w_{k+1}\right)} \phi\left(w_{k+1}\right) \frac{1}{\langle \sigma, X_{k+1}\rangle} dy_{k+1} \\
&= \int_{-\infty}^{\infty} \phi\left(y_{k+1}\right) dy_{k+1} = 1.
\end{aligned}$$

In the second equality above, note that the appropriate density to be used when taking the conditional expectation of λ_{k+1} is the density of w_{k+1}, since λ_{k+1} is ultimately a function of w_{k+1}, as y_{k+1} is itself a function of w_{k+1}. Integrating the entire standard normal density yields one.

So, we need to check only the numerator:

$$\begin{aligned}
\overline{P}\left(y_{k+1} \leq t\,|\Xi_k\right) &= E\left[\lambda_{k+1} I_{y_{k+1}\leq t}\,|\Xi_k\right] \\
&= \int_{-\infty}^{\infty} \frac{\langle \sigma, X_{k+1}\rangle \phi\left(y_{k+1}\right)}{\phi\left(w_{k+1}\right)} I_{y_{k+1}\leq t} \phi\left(w_{k+1}\right) \frac{1}{\langle \sigma, X_{k+1}\rangle} dy_{k+1} \\
&= \int_{-\infty}^{t} \phi\left(y_{k+1}\right) dy_{k+1} = \overline{P}\left(y_{k+1} \leq t\right),
\end{aligned}$$

which is the desired result. ∎

6.12 Proof of Lemma 3.4.12

Proof In order to prove lemma (1.4.12), it must be shown that under both P and $\overline{P}$, X has the Markov property and has both a matrix of transition probabilities A and an initial distribution p_0. Consider the initial distribution under the two probability measures first. The equality of the initial distributions of X under both probability measures follows as a direct consequence of a version of the conditional Bayes theorem:

$$\begin{aligned} E\left[X_0\right] &= \frac{\overline{E}\left[\Lambda_0^{-1} X_0 \left| \{\Omega, \emptyset\}\right.\right]}{\overline{E}\left[\Lambda_0^{-1} \left| \{\Omega, \emptyset\}\right.\right]} \\ &= \frac{\overline{E}\left[X_0\right]}{\overline{E}\left[1\right]} \\ &= \overline{E}\left[X_0\right] = p_0. \end{aligned}$$

To complete the proof, we need to consider the relation between the expectations $E\left[\langle X_k, e_j\rangle \left| \Xi_{k-1}\right.\right]$ and $\overline{E}\left[\langle X_k, e_j\rangle \left| \Xi_{k-1}\right.\right]$, as well as their values. The general idea for doing this is the same as in the discrete observations case. First, we will apply a version of the conditional Bayes theorem and prove that the denominator is equal to one. Then, we will evaluate the numerator.

$$\begin{aligned} E\left[\langle X_k, e_j\rangle \left| \Xi_{k-1}\right.\right] &= \frac{\overline{E}\left[\Lambda_k^{-1}\langle X_k, e_j\rangle \left| \Xi_{k-1}\right.\right]}{\overline{E}\left[\Lambda_k^{-1} \left| \Xi_{k-1}\right.\right]} \\ &= \frac{\Lambda_{k-1}^{-1}\overline{E}\left[\lambda_k^{-1}\langle X_k, e_j\rangle \left| \Xi_{k-1}\right.\right]}{\Lambda_{k-1}^{-1}\overline{E}\left[\lambda_k^{-1} \left| \Xi_{k-1}\right.\right]} \\ &= \frac{\overline{E}\left[\lambda_k^{-1}\langle X_k, e_j\rangle \left| \Xi_{k-1}\right.\right]}{\overline{E}\left[\lambda_k^{-1} \left| \Xi_{k-1}\right.\right]}. \end{aligned} \tag{6.8}$$

Now, examine the denominator in equation (6.8). Using the "tower" property of conditional expectations and substituting for w_k, we can rewrite it as:

$$\begin{aligned} \overline{E}\left[\lambda_k^{-1} \left| \Xi_{k-1}\right.\right] &= \overline{E}\left[\frac{\phi\left(\frac{y_k - \langle c, X_k\rangle}{\langle \sigma, X_k\rangle}\right)}{\phi\left(y_k\right)\langle \sigma, X_k\rangle} \left| \Xi_{k-1}\right.\right] \\ &= \overline{E}\left[\overline{E}\left[\frac{\phi\left(\frac{y_k - \langle c, X_k\rangle}{\langle \sigma, X_k\rangle}\right)}{\phi\left(y_k\right)\langle \sigma, X_k\rangle} \left| X_k, \Xi_{k-1}\right.\right] \left| \Xi_{k-1}\right.\right]. \end{aligned} \tag{6.9}$$

Consider the inner expectation in equation (6.9). It can be evaluated by integrating with respect to y_k:

$$\begin{aligned} \overline{E}\left[\frac{\phi\left(\frac{y_k - \langle c, X_k\rangle}{\langle \sigma, X_k\rangle}\right)}{\phi\left(y_k\right)\langle \sigma, X_k\rangle} \left| X_k, \Xi_{k-1}\right.\right] &= \frac{1}{\langle \sigma, X_k\rangle}\int_{\mathbb{R}} \frac{\phi\left(\frac{\xi - \langle c, X_k\rangle}{\langle \sigma, X_k\rangle}\right)}{\phi\left(\xi\right)}\phi\left(\xi\right) d\xi \\ &= \frac{1}{\langle \sigma, X_k\rangle}\int_{\mathbb{R}} \phi\left(\frac{\xi - \langle c, X_k\rangle}{\langle \sigma, X_k\rangle}\right) d\xi. \end{aligned}$$

We make the substitution $\varphi = \frac{\xi - \langle c, X_k \rangle}{\langle \sigma, X_k \rangle}$. Then, $d\xi = \langle \sigma, X_k \rangle d\varphi$, and the integral above becomes:

$$\begin{aligned} &= \frac{1}{\langle \sigma, X_k \rangle} \int_{\mathbb{R}} \phi(\varphi) \langle \sigma, X_k \rangle d\varphi \\ &= 1, \end{aligned}$$

since $\phi(\cdot)$ is the standard normal density.

Therefore, we are only interested in the numerator in equation (6.8):

$$E\left[\langle X_k, e_j \rangle \,|\Xi_{k-1}\right] = \overline{E}\left[\lambda_k^{-1} \langle X_k, e_j \rangle \,|\Xi_{k-1}\right]. \tag{6.10}$$

Conditioning twice in the same way as we did above, it becomes evident that:

$$\begin{aligned} \overline{E}\left[\lambda_k^{-1} \langle X_k, e_j \rangle \,|X_k, \Xi_{k-1}\right] &= \frac{\langle X_k, e_j \rangle}{\langle \sigma, X_k \rangle} \int_{\mathbb{R}} \phi(\varphi) \langle \sigma, X_k \rangle d\varphi \\ &= \langle X_k, e_j \rangle. \end{aligned} \tag{6.11}$$

Therefore, we have the following relation between conditional expectations under the two probability measures:

$$\begin{aligned} E\left[\langle X_k, e_j \rangle \,|\Xi_{k-1}\right] = \overline{E}\left[\lambda_k^{-1} \langle X_k, e_j \rangle \,|\Xi_{k-1}\right] &= \overline{E}\left[\langle X_k, e_j \rangle \,|\Xi_{k-1}\right] \quad (6.12) \\ &= \overline{E}\left[\langle X_k, e_j \rangle \,|X_{k-1}\right] \\ &= \langle A X_{k-1}, e_j \rangle. \quad \blacksquare \end{aligned}$$

6.13 Proof of Lemma 3.4.13

Proof First, note that $\overline{\Lambda}_k$ is Ξ_k-measurable. Therefore, we can write:

$$\overline{E}\left[\overline{\Lambda}_{k+1} \,|\Xi_k\right] = \overline{\Lambda}_k \overline{E}\left[\overline{\lambda}_{k+1} \,|\Xi_k\right].$$

Consequently, it is enough to show that $\overline{E}\left[\overline{\lambda}_{k+1} \,|\Xi_k\right] = 1$. This is almost immediately obvious when we substitute for $\overline{\lambda}_{k+1}$ and integrate with respect to y_{k+1}:

$$\begin{aligned} \overline{E}\left[\overline{\lambda}_{k+1} \,|\Xi_k\right] &= \overline{E}\left[\frac{\phi\left(\frac{y_{k+1} - \langle c, X_k \rangle}{\langle \sigma, X_k \rangle}\right)}{\langle \sigma, X_k \rangle \phi(y_{k+1})} \,|\Xi_k\right] \\ &= \int_{\mathbb{R}} \frac{\phi\left(\frac{\xi - \langle c, X_k \rangle}{\langle \sigma, X_k \rangle}\right)}{\langle \sigma, X_k \rangle \phi(\xi)} \phi(\xi)\, d\xi. \end{aligned}$$

As in the proof above, we make the change of variables $\varphi = \frac{\xi - \langle c, X_k \rangle}{\langle \sigma, X_k \rangle}$, so the variable of integration changes according to $d\xi = \langle \sigma, X_k \rangle d\varphi$, and we obtain:

$$\int_{\mathbb{R}} \frac{\phi\left(\frac{\xi - \langle c, X_k \rangle}{\langle \sigma, X_k \rangle}\right)}{\langle \sigma, X_k \rangle \phi(\xi)} \phi(\xi)\, d\xi = \int_{\mathbb{R}} \phi(\varphi)\, d\varphi = 1.$$

The result follows. $\blacksquare$

6.14 Proof of Lemma 3.4.14

Proof Let $f(\cdot) : \mathbb{R} \to \mathbb{R}$ be an integrable, Ξ_k-measurable function. Then, by a version of the conditional Bayes theorem, we have:

$$\begin{aligned} E\left[f\left(w_{k+1}\right)|\Xi_k\right] &= \frac{\overline{E}\left[\overline{\Lambda}_{k+1} f\left(w_{k+1}\right)|\Xi_k\right]}{\overline{E}\left[\overline{\Lambda}_{k+1}|\Xi_k\right]} \\ &= \overline{E}\left[\overline{\lambda}_{k+1} f\left(w_{k+1}\right)|\Xi_k\right], \end{aligned}$$

where the last equality comes from the fact that $\overline{E}\left[\overline{\lambda}_{k+1}|\Xi_k\right] = 1$, which was proven above. Therefore, substituting for w_{k+1} and $\overline{\lambda}_{k+1}$, we obtain:

$$\begin{aligned} E\left[f\left(w_{k+1}\right)|\Xi_k\right] &= \overline{E}\left[\overline{\lambda}_{k+1} f\left(w_{k+1}\right)|\Xi_k\right] \\ &= \overline{E}\left[\frac{\phi\left(\frac{y_{k+1}-\langle c, X_k\rangle}{\langle\sigma, X_k\rangle}\right)}{\langle\sigma, X_k\rangle\phi\left(y_{k+1}\right)} f\left(\frac{y_{k+1}-\langle c, X_k\rangle}{\langle\sigma, X_k\rangle}\right)|\Xi_k\right]. \end{aligned}$$

Since the $\{y_k\}$ are independent and identically distributed under $\overline{P}$, the subscripts become irrelevant. We evaluate the above expectation by integrating with respect to y and making the change of variables $\varphi = \frac{\xi-\langle c, X_k\rangle}{\langle\sigma, X_k\rangle}$, which implies that the variable of integration changes according to $d\xi = \langle\sigma, X_k\rangle d\varphi$. Thus, we obtain:

$$\int_{\mathbb{R}} \frac{\phi\left(\frac{\xi-\langle c, X_k\rangle}{\langle\sigma, X_k\rangle}\right)}{\langle\sigma, X_k\rangle\phi\left(\xi\right)} f\left(\frac{\xi-\langle c, X_k\rangle}{\langle\sigma, X_k\rangle}\right)\phi\left(\xi\right) d\xi = \int_{\mathbb{R}} \phi\left(\varphi\right) f\left(\varphi\right) d\varphi,$$

which completes the proof. ∎

6.15 Proof of Lemma 3.4.15

Proof The proof follows directly from the definitions of q_{k+1} and $\overline{\lambda}_{k+1}$:

$$\begin{aligned} q_{k+1} &= \overline{E}\left[\overline{\Lambda}_{k+1} X_{k+1}|\Upsilon_{k+1}\right] \\ &= \overline{E}\left[\overline{\Lambda}_k \frac{\phi\left(\frac{y_{k+1}-\langle c, X_k\rangle}{\langle\sigma, X_k\rangle}\right)}{\langle\sigma, X_k\rangle\phi\left(y_{k+1}\right)} X_{k+1}|\Upsilon_{k+1}\right] \\ &= \sum_{j=1}^{N} \overline{E}\left[\overline{\Lambda}_k\langle X_{k+1}, e_j\rangle|\Upsilon_k\right]\psi_j\left(y_{k+1}\right) e_j \\ &= \sum_{j=1}^{N}\langle A q_k, e_j\rangle\psi_j\left(y_{k+1}\right) e_j \\ &= B\left(y_{k+1}\right) A q_k. \end{aligned}$$

Perhaps the only remark worth noting in the above proof is that the third equality follows from the fact, that under the probability measure $\overline{P}$, the observation process $\{y_k\}$ is a sequence of i.i.d. random variables. Hence, when taking conditional expectations, conditioning on Υ_k and Υ_{k+1} will yield the same result, since knowing the value of y_{k+1} does not bring any useful additional information about the future.

Bibliography

Aggoun, L. & Elliott, R. J. (2004). *Measure Theory and Filtering: Introduction and Applications*, Cambridge University Press.

Ambachtsheer, K. P. (1987). Pension fund asset allocation: In defense of a 60/40 equity/debt asset mix, *Financial Analysts Journal* **43**: 14–24.

Ando, A. & Modigliani, F. (1957). The life cycle hypothesis of saving: Aggregate implications and tests, *American Economic Review* **53**: 55–84.

Ang, A. & Bekaert, G. (2002). International asset allocation with regime shifts, *Review of Financial Studies* **15**: 1137–1187.

Antoci, A. (1995). Stock market dynamics with institutional trading, *Decisions in Economics and Finance* **18**: 143–151.

Anufriev, M. & Dindo, P. (2010). Wealth-driven selection in a financial market with heterogeneous agents, *Journal of Economic Behavior & Organization* **73**: 327–358.

Apilado, V. P. (1972). Pension funds, personal savings, and economic growth, *Journal of Risk and Insurance* **39**: 397–404.

Arena, M. (2008). Does insurance market activity promote economic growth? a cross-country study for industrialized and developing countries, *Journal of Risk and Insurance* **75**: 921–946.

Ash, R. B. & Doléans-Dade, C. (2000). *Probability and Measure Theory*, 2nd edn, Academic Press.

Ausloos, M., Miśkiewicz, J. & Sanglier, M. (2004). The durations of recession and prosperity: Does their distribution follow a power or an exponential law, *Physica A* **339**(3-4): 548–558.

Azcue, P. & Muler, N. (2009). Optimal investment strategy to minimize the ruin probability of an insurance company under borrowing constraints, *Insurance: Mathematics and Economics* **44**: 26–34.

Badrinath, S. G., Kale, J. R. & Noe, T. H. (1995). Of shepherds, sheep, and the cross-autocorrelations in equity returns, *Review of Financial Studies* **8**: 401–430.

Bai, L. & Guo, J. (2008). Optimal proportional reinsurance and investment with multiple risky assets and no-shorting constraint, *Insurance: Mathematics and Economics* **42**: 968–975.

Bank, P. & Baum, D. (2004). Hedging and portfolio optimization in financial markets with a large trader, *Mathematical Finance* **14**: 1–18.

Banz, R. (1981). The relationship between return and market value of common stocks, *Journal of Financial Economics* **9**: 1–18.

Baum, L. E. & Petrie, T. (1966). Statistical inference for probabilistic functions of finite state markov chains, *The Annals of Mathematical Statistics* **37**(6): 1554–1563.

Berry-Stolzle, T. R. (2008). Evaluating liquidation strategies for insurance companies, *Journal of Risk and Insurance* **75**: 207–230.

Björk, T. (2009). *Arbitrage Theory in Continuous Time*, 3rd edn, Oxford University Press.

Black, F. & Perold, A. F. (1992). Theory of constant proportion portfolio insurance, *Journal of Economic Dynamics and Control* **16**: 403–426.

Blake, D. (1998). Pension schemes as options on pension fund assets: Implications for pension fund management, *Insurance: Mathematics and Economics* **23**: 263–286.

Blake, D. (2003). *Pension Schemes and Pension Funds in the United Kingdom*, 2nd edn, Oxford University Press.

Blake, D. (2006). *Pension Economics*, John Wiley & Sons.

Blake, D., Lehmann, B. N. & Timmermann, A. G. (1999). Asset allocation dynamics and pension fund performance, *Journal of Business* **72**: 429–461.

Blume, L. & Easley, D. (1992). Evolution and market behavior, *Journal of Economic Theory* **58**: 9–40.

Blume, L. & Easley, D. (2006). If you're so smart, why aren't you rich? belief selection in complete and incomplete markets, *Econometrica* **74**(4): 929–966.

Bochkanov, S. (2012). The alglib open source project, `http://www.alglib.net/`.

Bodie, Z. (1989). Pensions as retirement income insurance, *NBER Working Paper W2917*.

Bodie, Z. (1990). Managing pension and retirement assets: An international perspective, *Journal of Financial Services Research* **4**: 419–460.

Bodie, Z. (1991). Shortfall risk and pension fund asset management, *Financial Analysts Journal* **47**: 57–61.

Bogle, J. C. (1994). *Bogle on Mutual Funds: New Perspectives for the Intelligent Investor*, Dell.

Bohl, M. T. & Brzeszczynski, J. (2006). Do institutional investors destabilize stock prices? evidence from an emerging market, *Journal of International Financial Markets, Institutions and Money* **16**: 370–383.

Brinson, G. P., Hood, L. R. & Beebower, G. L. (1995). Determinants of portfolio performance, *Financial Analysts Journal* **51**: 133–139.

Browne, S. (1997). Survival and growth with a liability: Optimal portfolio strategies in continuous time, *Mathematics of Operations Research* **22**: 468–493.

Catalan, M., Impavido, G. & Musalem, A. R. (2000). Contractual savings or stock market developments: Which leads?, *World Bank Policy Research Working Paper* **2421**.

Chakravarty, S. (2001). Stealth-trading: Which traders' trades move stock prices, *Journal of Financial Economics* **61**: 289–307.

Chan, L. K. C. & Lakonishok, J. (1995). The behavior of stock prices around institutional trades, *Journal of Finance* **50**: 1147–1174.

Chiarella, C., Dieci, R. & He, X. (2007). Heterogeneous expectations and speculative behavior in a dynamic multi-asset framework, *Journal of Economic Behavior and Organization* **62**: 408–427.

Chiarella, C., Dieci, R. & He, X. (2009). *Handbook of Financial Markets: Dynamics and Evolution*, North-Holland, chapter Heterogeneity, Market Mechanisms, and Asset Price Dynamics, pp. 277–345.

Chiyachantana, C. N., Jain, P. K., Jiang, C. & Wood, R. A. (2004). International evidence on institutional trading behavior and price impact, *Journal of Finance* **59**: 869–898.

Davis, E. P. (1996). The role of institutional investors in the evolution of financial structure and behaviour, *Financial Markets Group Special Papers, London School of Economics* .

Davis, E. P. (2000). Pension funds, financial intermediation and the new financial landscape, *Pensions Institute Discussion Paper PI-0010, City University* .

Davis, E. P. & Hu, Y. W. (2004). Is there a link between pension fund assets and economic growth? a cross-country study, *Brunel University Working Paper* **04-23**.

Davis, E. P. & Steil, B. (2004). *Institutional Investors*, MIT Press.

Detemple, J. & Rindisbacher, M. (2008). Dynamic asset liability management with tolerance for limited shortfalls, *Insurance: Mathematics and Economics* **43**: 281–294.

Diamond, P. (1965). National debt in a neoclassical growth model, *American Economic Review* **55**: 1126–1150.

Dixit, A. K. & Pindyck, R. S. (1994). *Investment under Uncertainty*, Princeton University Press.

Doob, J. L. (1953). *Stochastic Processes*, John Wiley & Sons.

Dorfman, M. S. (2004). *Introduction to Risk Management and Insurance*, 8th edn, Prentice Hall.

Dufour, A. & Engle, R. F. (1999). Time and price impact of a trade, *University of California at San Diego Discussion Paper* **99-15**.

Durland, J. M. & McCurdy, T. H. (1994). Duration-dependent transitions in a markov model of u.s. gnp growth, *Journal of Business and Economic Statistics* **12**(3): 279–288.

Dybvig, P. H. (1995). Dusenberry's ratcheting of consumption: Optimal dynamic consumption and investment given intolerance for any decline in standard of living, *Review of Economic Studies* **62**: 287–313.

Ehrentreich, N. (2008). *Agent-Based Modeling: The Santa Fe Institute Artificial Stock Market Model Revisited (Lecture Notes in Economics and Mathematical Systems)*, Academic Press.

Elliott, R. J., Aggoun, L. & Moore, J. B. (1995). *Hidden Markov Models: Estimation and Control*, Springer.

Evstigneev, I., Hens, T. & Schenk-Hoppé, K. R. (2002). Market selection of financial trading strategies: Global stability, *Mathematical Finance* **12**: 329–339.

Evstigneev, I. V., Hens, T. & Schenk-Hoppé, K. R. (2006). Evolutionary stable stock markets, *Economic Theory* **27**: 449–468.

Evstigneev, I. V., Hens, T. & Schenk-Hoppé, K. R. (2008). Globally evolutionarily stable portfolio rules, *Journal of Economic Theory* **140**: 197–228.

Evstigneev, I. V., Hens, T. & Schenk-Hoppé, K. R. (2009). *Handbook of Financial Markets: Dynamics and Evolution*, North-Holland, chapter Evolutionary Finance, pp. 507–566.

Feller, W. (1951). Two singular diffusion problems, *The Annals of Mathematics* **54**(1): 173–182.

Fischer, S. (1983). *Financial Aspects of the United States Pension System*, University of Chicago Press, chapter Investing for the Short and the Long Term, pp. 153–175.

Fletcher, R. (2000). *Practical Methods of Optimization*, 2nd edn, John Wiley & Sons.

Fraser, A. M. (2009). *Hidden Markov Models and Dynamical Systems*, Society for Industrial Mathematics.

Gaunersdorfer, A. (2000). Endogenous fluctuations in a simple asset pricing model with heterogeneous beliefs, *Journal of Economic Dynamics and Control* **24**: 799–831.

Gibson, R. C. (2008). *Asset Allocation: Balancing Financial Risk*, 4 edn, McGraw-Hill.

Giorgi, E. D. (2008). Evolutionary portfolio selection with liquidity shocks, *Journal of Economic Dynamics and Control* **32**: 1088–1119.

Goldfarb, D. & Idnani, A. (1983). A numerically stable dual method for solving strictly convex quadratic programs, *Mathematical Programming* **27**: 1–33.

Goldman, M. B. (1979). Anti-diversification, or optimal programs for infrequently revised portfolios, *Journal of Finance* **34**: 505–516.

Gompers, P. A. & Metrick, A. (2001). Institutional investors and equity prices, *Quarterly Journal of Economics* **116**: 229–259.

Graham, L. & Xie, X. (2007). *Handbook of International Insurance*, Springer-Verlag, chapter The United States Insurance Market: Characteristics and Trends, pp. 25–137.

Hainaut, D. & Devolder, P. (2007). Management of a pension fund under mortality and financial risks, *Insurance: Mathematics and Economics* **41**: 134–155.

Hamilton, J. D. (1989). A new approach to the economic analysis of nonstationary time series and the business cycle, *Econometrica* **57**(2): 357–384.

Hardwick, P. & Guirguis, M. (2007). *Handbook of International Insurance*, Springer-Verlag, chapter The UK Insurance Industry - Structure and Performance, pp. 205–235.

Harichandra, K. & Thangavelu, S. M. (2004). Institutional investors, financial sector development and economic growth in oecd countries, *National University of Singapore Working Paper* **0405**.

Harrison, J. M. & Sharpe, W. F. (1983). *Financial Aspects of the United States Pension System*, University of Chicago Press, chapter Optimal Funding and Asset Allocation Rules for Defined-Benefit Pension Plans, pp. 91–105.

He, H. & Mamaysky, H. (2005). Dynamic trading policies with price impact, *Journal of Economic Dynamics and Control* **29**: 891–930.

Hens, T. & Schenk-Hoppé, K. R. (2005). Evolutionary stability of portfolio rules in incomplete markets, *Journal of Mathematical Economics* **41**: 43–66.

Higham, D. (2001). An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Review* **43**(3): 525–546.

Higham, D. & Mao, X. (2005). Convergence of monte carlo simulations involving the mean-reverting square root process, *Journal of Computational Finance* **8**(3): 35–62.

Hommes, C. H. & Wagener, F. O. (2009). *Handbook of Financial Markets: Dynamics and Evolution*, North-Holland, chapter Complex Evolutionary Systems in Behavioral Finance, pp. 217–276.

Ingersoll, J. E. (1992). Optimal consumption and portfolio rules with intertemporally dependent utility of consumption, *Journal of Economic Dynamics and Control* **16**: 681–712.

Itô, K. (1951). On stochastic differential equations, *Memoirs of the American Mathematical Society* **4**: 1–51.

Jazwinski, A. (1970). *Stochastic Processes and Filtering Theory*, Academic Press.

Kelly, J. (1956). A new interpretation of information rate, *Bell System Technical Journal* **35**: 917–926.

Kim, C. & Nelson, C. R. (1999). Friedman's plucking model of business fluctuations: Tests and estimates of permanent and transitory components, *Journal of Money, Credit and Banking* **31**(3): 317–334.

Kloeden, P. E. & Platen, E. (1999). *Numerical Solution of Stochastic Differential Equations*, Springer.

Kruse, R. L. & Ryba, A. J. (2000). *Data Structures and Program Design in C++*, Prentice Hall.

Kugler, M. & Ofoghi, R. (2005). Does insurance promote economic growth? evidence from the uk, *Money Macro and Finance Research Group Conference* .

Kuhn, H. W. & Tucker, A. W. (1951). Nonlinear programming, *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* pp. 481–492.

Kwok, Y. (2008). *Mathematical Models of Financial Derivatives*, 2nd edn, Springer.

Lakonishok, J., Shleifer, A. & Vishny, R. W. (1992). The impact of institutional trading on stock prices, *Journal of Financial Economics* **32**: 23–43.

Latané, H. A. (1959). Criteria for choice among risky ventures, *Journal of Political Economy* **67**: 144–155.

Leland, H. E. (1980). Who should buy portfolio insurance?, *Journal of Finance* **35**: 581–594.

Levine, R. (2005). *Handbook of Economic Growth (Volume 1A)*, North-Holland, chapter Finance and Growth: Theory and Evidence, pp. 865–923.

London, I. F. S. (2008). Insurance update, *IFSL Research* .

Lux, T. (2009). *Handbook of Financial Markets: Dynamics and Evolution*, North-Holland, chapter Stochastic Behavioral Asset Pricing Models and the Stylized Facts, pp. 161–211.

MacLean, L. C., Sanegre, R., Zhao, Y. & Ziemba, W. T. (2004). Capital growth with security, *Journal of Economic Dynamics and Control* **28**: 937–954.

MacLean, L. C., Ziemba, W. T. & Blazenko, G. (1992). Growth versus security in dynamic investment analysis, *Management Science* **38**: 1562–1585.

Marsaglia, G. (1964). Generating a variable from the tail of the normal distribution, *Technometrics* **6**: 101–102.

Marsaglia, G. & Tsang, W. W. (2000). The ziggurat method for generating random variables, *Journal of Statistical Software* **5**.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous time case, *Review of Economics and Statistics* **51**: 239–246.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory* **3**: 373–413.

Merton, R. C. (1972). An analytic derivation of the efficient portfolio frontier, *Journal of Financial and Quantitative Analysis* **7**: 1851–1872.

Merton, R. C. & Samuelson, P. (1973). Fallacy of the log-normal approximation to optimal portfolio decision-making over many periods, *Journal of Financial Economics* **1**: 67–94.

Mossin, J. (1968). Optimal multiperiod portfolio policies, *Journal of Business* **41**: 215–229.

Murata, Y. (1977). *Mathematics for Stability and Optimization of Economic Systems*, Academic Press.

Nunen, A. V. (2008). *Fiduciary Management - Blueprint for Pension Fund Excellence*, John Wiley & Sons.

Øksendal, B. K. (2002). *Stochastic Differential Equations: An Introduction with Applications*, 5th edn, Springer.

Outreville, F. (1990). The economic significance of insurance markets in developing countries, *Journal of Risk and Insurance* **57**: 487–498.

Palczewski, J. & Schenk-Hoppé, K. R. (2010a). From discrete to continuous time evolutionary finance models, *Journal of Economic Dynamics and Control* **34**: 913–931.

Palczewski, J. & Schenk-Hoppé, K. R. (2010b). Market selection of constant proportions investment strategies in continuous time, *Journal of Mathematical Economics* **46**: 248–266.

Palmer, R. G., Arthur, W. B., Holland, J. H. & LeBaron, B. (1999). An artificial stock market, *Artificial Life Robotics* **3**: 27–31.

Palmer, R. G., Arthur, W. B., LeBaron, B. & Taylor, P. (1994). Artficial economic life: A simple model of a stockmarket, *Physica D* **75**: 264–274.

Rabiner, L. R. (1989). A tutorial on hidden markov models and selected applications in speech recognition, *Proceedings of the IEEE* **77**(2): 257–286.

Reisen, H. & Williamson, J. (1994). Pension funds, capital controls and macroeconomic stability, *OECD Working Paper* **98**.

Saar, G. (2001). Price impact asymmetry of block trades: An institutional trading explanation, *Review of Financial Studies* **14**: 1153–1181.

Samuelson, P. (1958). An exact consumption-loan model of interest with or without the social contrivance of money, *Journal of Political Economy* **66**: 467–482.

Samuelson, P. (1969). Lifetime portfolio selection by dynamic stochastic programming, *Review of Economics and Statistics* **51**: 239–246.

Samuelson, P. (1971). The "fallacy" of maximizing the geometric mean in long sequences of investing or gambling, *Proceedings of the National Academy of Sciences* **68**: 2493–2496.

Shreve, S. E. (2004). *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer.

Sias, R. W. & Starks, L. T. (1997). Return autocorrelation and institutional investors, *Journal of Financial Economics* **46**: 103–131.

Sias, R. W., Starks, L. T. & Titman, S. (2001). The price impact of institutional trading, *Washington State University Working Paper* .

Skipper, H. (1997). Foreign insurers in emerging markets: Issues and concerns, *Center for Risk Management and Insurance Occasional Paper* **97-2**.

Stokey, N. L., Lucas, R. E. & Prescott, E. C. (1989). *Recursive Methods in Economic Dynamics*, Harvard University Press.

Thaler, R. H. & Williamson, J. P. (1994). College and university endowment funds: Why not 100 percent equities?, *Journal of Portfolio Management* **21**: 27–38.

Thorp, E. O. (2006). *Handbook of Asset and Liability Management Volume 1: Theory and Methodology*, North-Holland, chapter The Kelly Criterion in Blackjack, Sports Betting, and the Stock Market, pp. 386–418.

Vath, L. V., Mnif, M. & Pham, H. (2007). A model of optimal portfolio selection under liquidity risk and price impact, *Finance and Stochastics* **11**: 51–90.

Vaughan, E. J. & Vaughan, T. M. (2002). *Fundamentals of Risk and Insurance*, 9th edn, John Wiley & Sons.

Vittas, D. (1998). Institutional investors and security markets: Which comes first?, *World Bank Policy Research Working Paper* **2032**.

Walther, B. R. (1997). Investor sophistication and market earnings expectations, *Journal of Accounting Research* **35**: 157–179.

Ward, D. & Zurbruegg, R. (2000). Does insurance promote economic growth? evidence from oecd countries, *Journal of Risk and Insurance* **67**: 489–506.

Webb, I., Grace, M. F. & Skipper, H. (2002). The effect of banking and insurance on the growth of capital and output, *Center for Risk Management and Insurance Working Paper* **02**.

Wolfe, P. (1959). The simplex method for quadratic programming, *Econometrica* **27**: 382–398.

Zhang, X., Zhang, K. & Yu, X. (2009). Optimal proportional reinsurance and investment with transaction costs: Maximizing the terminal wealth, *Insurance: Mathematics and Economics* **44**: 473–478.

Zhou, C., Wu, C., Zhang, S. & Huang, X. (2008). An optimal insurance strategy for an individual under an intertemporal equilibrium, *Insurance: Mathematics and Economics* **42**: 255–260.

Ziemba, W. T. (2005). *The Best of Wilmott: Volume 2*, John Wiley and Sons, chapter Good and Bad Properties of the Kelly Criterion, pp. 65–72.